HW1 solution

Problem 1. The answer in not unique.

(1) The probability interpretation of quantum mechanics, which is also called the Copenhagen interpretation, is one of the earliest and most commonly taught interpretations of quantum mechanics. It holds that quantum mechanics does not yield a description of an objective reality but deals only with probabilities of observing, or measuring, various aspects of energy quanta, entities that fit neither the classical idea of particles nor the classical idea of waves. According to the interpretation, the act of measurement causes the set of probabilities to immediately and randomly assume only one of the possible values. This feature of the mathematics is known as wavefunction collapse. The essential concepts of the interpretation were devised by Niels Bohr, Werner Heisenberg and others in the years 1924–27.

Suppose we have a physical observable described by Hermitian operator A, the corresponding eigenvalues are n_1, n_2, n_3, $\cdots$, and the corresponding eigenstates are $|n_1\rangle$, $|n_2\rangle$, $|n_3\rangle$, $\cdots$. When we perform a measurement on the state

$$|\psi\rangle = \sum_i c_i |n_i\rangle, \tag{1}$$

the possible measurement outcomes are just n_1, n_2, n_3, $\cdots$, and the state will collapse into the corresponding eigenstates $|n_1\rangle$, $|n_2\rangle$, $|n_3\rangle$, $\cdots$. The probability of finding state $|n_i\rangle$ is

$$P_i = |\langle n_i|\psi\rangle|^2 = |c_i|^2. \tag{2}$$

The mean value of the results of the repeated measurements of A is

$$\langle\psi|A|\psi\rangle = \sum_i P_i n_i. \tag{3}$$

(2) The relation $\Delta x \Delta p \geqslant \frac{\hbar}{2}$ can be regarded as a fundamental limitation on the possibility of preparing a quantum state $|\psi\rangle$ to have statistical dispersions that violate the inequality. More information about uncertainty relation can be found on Wiki or textbook.

(3) The de Broglie relation shows the concept of matter waves or de Broglie waves and reflects the wave–particle duality of matter. It can be expressed as

$$\lambda = \frac{h}{p}, \tag{4}$$

where λ is the wavelength of the matter, p is the momentum, and h is the Planck constant. For the pingpong ball, its wavelength is

$$\lambda = \frac{h}{mv} = \frac{6.626 \times 10^{-34} m^2 kg/s}{2 \times 10^{-3} kg \times 10 m/s} = 3.313 \times 10^{-32} m. \tag{5}$$

That is why it always difficult for us to observe quantum effect of macroscopic objects, because their wavelength is so small.

Problem 2.

(a) Measurement results should be the eigenvalues $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$. The probabilities are

$$P_+ = |\langle + | \psi \rangle|^2 = \frac{9}{34}, \tag{6}$$

for state $|+\rangle$, and

$$P_- = |\langle - | \psi \rangle|^2 = \frac{25}{34}, \tag{7}$$

for state $|-\rangle$.

(b) According to the measurement theory of quantum mechanics, the state after measurement collapses into $|-\rangle$ immediately. Then, if we perform a subsequent measurement , we have 100% probability to obtain $|-\rangle$ again.

(c) Performing the measurement to x-component will result in the state "$|-\rangle_z$" collapse into the eigenstates of S_x. Since

$$|-\rangle_z = \frac{1}{\sqrt{2}}(|+\rangle_x - |-\rangle_x), \tag{8}$$

The probability of finding $|+\rangle_x$ is

$$|_x\langle + | - \rangle_z|^2 = \frac{1}{2} \tag{9}$$

The probability of finding $|-\rangle_x$ is

$$|_x\langle - | - \rangle_z|^2 = \frac{1}{2} \tag{10}$$

(d) See the figure below

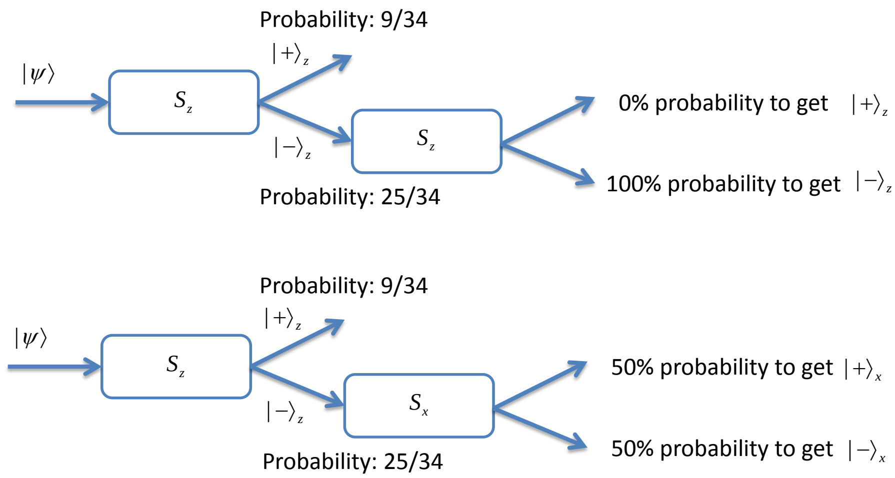

Figure 1: diagram of sequential S-G measurements.

HW2 solution

Problem 4.

Write down the Hamiltonian in matrix form in the basis of $\{|1\rangle, |2\rangle\}$ as

$$H = E \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{1}$$

The solve the Secular equation

$$det(H - \lambda I) = 0 \tag{2}$$

$$\begin{vmatrix} E - \lambda & E \\ E & -E - \lambda \end{vmatrix} = 0 \tag{3}$$

$$\lambda^2 - E^2 - E^2 = 0 \tag{4}$$

$$\lambda_1 = \sqrt{2}E, \quad \lambda_2 = -\sqrt{2}E \tag{5}$$

Then we have

$$E \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \sqrt{2}E \begin{bmatrix} a \\ b \end{bmatrix} \tag{6}$$

So, the eigenstate corresponding to $\lambda_1 = \sqrt{2}E$ is

$$|\psi_1\rangle = \frac{1}{\sqrt{2 + 2\sqrt{2}}}[(1 + \sqrt{2})|1\rangle + |2\rangle] \tag{7}$$

Similarly, the eigenstate corresponding to $\lambda_2 = -\sqrt{2}E$ is

$$|\psi_2\rangle = \frac{1}{\sqrt{2 - 2\sqrt{2}}}[(1 - \sqrt{2})|1\rangle + |2\rangle] \tag{8}$$

HW3 solution

1. (a) The coefficients a_k $(k = 0, 1, 2, 3)$ are

$$2a_k = \mathrm{Tr}(\sigma_k X), \tag{1}$$

where $\sigma_0 = I$, σ_k $(k = 1, 2, 3)$ are Pauli matrices.

Proof:

$$\begin{aligned} \mathrm{Tr}(IX) &= \mathrm{Tr}(a_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3) \\ &= 2a_0, \end{aligned} \tag{2}$$

because $\mathrm{Tr}(\sigma_k) = 0$ $(k = 1, 2, 3)$.

$$\begin{aligned} \mathrm{Tr}(\sigma_1 X) &= \mathrm{Tr}[\sigma_1(a_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)] \\ &= \mathrm{Tr}[a_0\sigma_1 + a_1 + ia_2\sigma_3 - ia_3\sigma_2)] \\ &= 2a_1. \end{aligned} \tag{3}$$

Similarly,

$$\mathrm{Tr}(\sigma_2 X) = 2a_2, \tag{4}$$

$$\mathrm{Tr}(\sigma_3 X) = 2a_3. \tag{5}$$

End of proof.

(b) Use the results in (a), we have

$$\begin{aligned} a_0 &= \frac{1}{2}\mathrm{Tr}(X) \\ &= \frac{1}{2}\mathrm{Tr}\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \\ &= \frac{1}{2}(X_{11} + X_{22}), \end{aligned} \tag{6}$$

$$\begin{aligned} a_1 &= \frac{1}{2}\mathrm{Tr}(\sigma_1 X) \\ &= \frac{1}{2}\mathrm{Tr}\left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}\right\} \\ &= \frac{1}{2}\mathrm{Tr}\begin{bmatrix} X_{21} & X_{22} \\ X_{11} & X_{12} \end{bmatrix} \\ &= \frac{1}{2}(X_{12} + X_{21}), \end{aligned} \tag{7}$$

$$\begin{aligned}
a_2 &= \frac{1}{2}\mathrm{Tr}(\sigma_2 X) \\
&= \frac{1}{2}\mathrm{Tr}\left\{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}\right\} \\
&= \frac{1}{2}\mathrm{Tr}\begin{bmatrix} -iX_{21} & -iX_{22} \\ iX_{11} & iX_{12} \end{bmatrix} \\
&= \frac{i}{2}(X_{12} - X_{21}), \qquad (8)
\end{aligned}$$

$$\begin{aligned}
a_3 &= \frac{1}{2}\mathrm{Tr}(\sigma_3 X) \\
&= \frac{1}{2}\mathrm{Tr}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}\right\} \\
&= \frac{1}{2}\mathrm{Tr}\begin{bmatrix} X_{11} & X_{12} \\ -X_{21} & -X_{22} \end{bmatrix} \\
&= \frac{1}{2}(X_{11} - X_{22}). \qquad (9)
\end{aligned}$$

2. (a) Choose any complete orthonormal basis $\{|n\rangle\}$, Then $\mathrm{Tr}A = \sum_n \langle n|A|n\rangle$, therefore,

$$\begin{aligned}
\mathrm{Tr}(XY) &= \sum_n \langle n|XY|n\rangle \\
&= \sum_{mn} \langle n|X|m\rangle\langle m|Y|n\rangle \\
&= \sum_{mn} \langle m|Y|n\rangle\langle n|X|m\rangle \\
&= \sum_m \langle m|YX|m\rangle \\
&= \mathrm{Tr}(YX). \qquad (10)
\end{aligned}$$

(b) From the original definition in mathematics, we have two following two mean values are equal

$$\langle A\psi, \phi\rangle = \langle \psi, A^\dagger \phi\rangle \qquad (11)$$

where ψ and ϕ are vectors in Hilbert space, and A can be any operators. According to this, we have

$$\begin{aligned}\langle (XY)^\dagger\psi,\phi\rangle &= \langle\psi,[(XY)^\dagger]^\dagger\phi\rangle \\ &= \langle\psi,XY\phi\rangle \\ &= \langle X^\dagger\psi,Y\phi\rangle \\ &= \langle Y^\dagger X^\dagger\psi,\phi\rangle. \end{aligned} \tag{12}$$

So,

$$(XY)^\dagger = Y^\dagger X^\dagger. \tag{13}$$

(c) Suppose the eigenvalues of A are λ_n with corresponding eigenvectors $|n\rangle$, i.e.,

$$A|n\rangle = \lambda_n|n\rangle. \tag{14}$$

Then, any function of operator acting on the eigenstate gives

$$f(A)|n\rangle = \sum_m \frac{f^{(m)}(a)}{m!}(A-a)^m|n\rangle, \tag{15}$$

taking the expansion point $a=0$, then

$$\begin{aligned} f(A)|n\rangle &= \sum_m \frac{f^{(m)}(0)}{m!}A^m|n\rangle \\ &= \sum_m \frac{f^{(m)}(0)}{m!}\lambda_n^m|n\rangle \\ &= f(\lambda_n)|n\rangle. \end{aligned}$$

Actually, this is the general proof for arbitrary function, $\exp[if(A)]$ is just a special case. Now we compute the matrix elements of it as

$$\begin{aligned}\exp[if(A)] &= \sum_n \exp[if(A)]|n\rangle\langle n| \\ &= \sum_n \exp[if(\lambda_n)]|n\rangle\langle n|. \end{aligned} \tag{16}$$

3. (1) Since

$$\psi_1 = \langle u_1|\psi\rangle = c_1 \tag{17}$$

$$\psi_2 = \langle u_2|\psi\rangle = c_2 \tag{18}$$

then,

$$|\psi\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{19}$$

(2) Calculate every matrix elements as

$$\langle u_1|S_z|u_1\rangle = \frac{\hbar}{2}\langle u_1|u_1\rangle = \frac{\hbar}{2} \tag{20}$$

$$\langle u_1|S_z|u_2\rangle = -\frac{\hbar}{2}\langle u_1|u_2\rangle = 0 \tag{21}$$

$$\langle u_2|S_z|u_1\rangle = \frac{\hbar}{2}\langle u_2|u_1\rangle = 0 \tag{22}$$

$$\langle u_2|S_z|u_2\rangle = -\frac{\hbar}{2}\langle u_2|u_2\rangle = -\frac{\hbar}{2} \tag{23}$$

So, the matrix form is

$$S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{24}$$

(3) Since

$$\psi_1 = \langle v_1|\psi\rangle = \frac{1}{\sqrt{2}}c_1 - \frac{1}{\sqrt{2}}c_2 \tag{25}$$

$$\psi_2 = \langle v_2|\psi\rangle = \frac{1}{\sqrt{2}}c_1 + \frac{1}{\sqrt{2}}c_2 \tag{26}$$

then,

$$|\psi\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}}c_1 - \frac{1}{\sqrt{2}}c_2 \\ \frac{1}{\sqrt{2}}c_1 + \frac{1}{\sqrt{2}}c_2 \end{bmatrix} \tag{27}$$

Then, we calculate matrix elements for S_z as

$$\begin{aligned} \langle v_1|S_z|v_1\rangle &= \frac{1}{2}(\langle u_1| - \langle u_2|)S_z(|u_1\rangle - |u_2\rangle) \\ &= \frac{1}{2}(\langle u_1| - \langle u_2|)\frac{\hbar}{2}(|u_1\rangle + |u_2\rangle) \\ &= 0 \end{aligned} \tag{28}$$

$$\begin{aligned} \langle v_1|S_z|v_2\rangle &= \frac{1}{2}(\langle u_1| - \langle u_2|)S_z(|u_1\rangle + |u_2\rangle) \\ &= \frac{1}{2}(\langle u_1| - \langle u_2|)\frac{\hbar}{2}(|u_1\rangle - |u_2\rangle) \\ &= \frac{\hbar}{2} \end{aligned} \tag{29}$$

$$\begin{aligned}
\langle v_2|S_z|v_1\rangle &= \frac{1}{2}(\langle u_1| + \langle u_2|)S_z(|u_1\rangle - |u_2\rangle) \\
&= \frac{1}{2}(\langle u_1| + \langle u_2|)\frac{\hbar}{2}(|u_1\rangle + |u_2\rangle) \\
&= \frac{\hbar}{2}
\end{aligned} \tag{30}$$

$$\begin{aligned}
\langle v_2|S_z|v_2\rangle &= \frac{1}{2}(\langle u_1| + \langle u_2|)S_z(|u_1\rangle + |u_2\rangle) \\
&= \frac{1}{2}(\langle u_1| + \langle u_2|)\frac{\hbar}{2}(|u_1\rangle - |u_2\rangle) \\
&= 0
\end{aligned} \tag{31}$$

So, the matrix form in this basis is

$$S_z = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{32}$$

From the relation

$$|v_1\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle - |u_2\rangle) \tag{33}$$

$$|v_2\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle) \tag{34}$$

we can see

$$|v_i\rangle = \sum_j U_{ij}|u_j\rangle \tag{35}$$

where the transformation matrix is

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{36}$$

HW4 solution

I. PART I

1. (1) Let S_z acting on $|+\rangle$ and $|-\rangle$

$$\begin{aligned} S_z|+\rangle &= \frac{\hbar}{2}[|+\rangle\langle+| - |-\rangle\langle-|]|+\rangle \\ &= \frac{\hbar}{2}[|+\rangle\langle+|+\rangle - |-\rangle\langle-|+\rangle] \\ &= \frac{\hbar}{2}|+\rangle \end{aligned} \tag{1}$$

$$\begin{aligned} S_z|+\rangle &= \frac{\hbar}{2}[|+\rangle\langle+| - |-\rangle\langle-|]|-\rangle \\ &= \frac{\hbar}{2}[|+\rangle\langle+|-\rangle - |-\rangle\langle-|-\rangle] \\ &= -\frac{\hbar}{2}|-\rangle \end{aligned} \tag{2}$$

So, $|+\rangle$ is the eigenstate of S_z with corresponding eigenvalue $\frac{\hbar}{2}$; $|-\rangle$ is the eigenstate of S_z with corresponding eigenvalue $-\frac{\hbar}{2}$.

(2) Compute the matrix elements as

$$\langle+|I|+\rangle = \langle+|(|+\rangle\langle+| + |-\rangle\langle-|)|+\rangle = 1 \tag{3}$$

$$\langle+|I|-\rangle = \langle+|(|+\rangle\langle+| + |-\rangle\langle-|)|-\rangle = 0 \tag{4}$$

$$\langle-|I|+\rangle = \langle-|(|+\rangle\langle+| + |-\rangle\langle-|)|+\rangle = 0 \tag{5}$$

$$\langle-|I|-\rangle = \langle-|(|+\rangle\langle+| + |-\rangle\langle-|)|-\rangle = 1 \tag{6}$$

Therefore the matrix form of I is

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7}$$

Thus, we show explicitly that I is the identity operator.

(3) In matrix form [see (4)],

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{8}$$

Solving the Secular equation, we can find the eigenvalues and eigenvectors as

$$\lambda_1 = \frac{\hbar}{2} \tag{9}$$

with the corresponding eigenvector

$$|x_1\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \tag{10}$$

and

$$\lambda_2 = -\frac{\hbar}{2} \tag{11}$$

with the corresponding eigenvector

$$|x_2\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \tag{12}$$

Similarly, in matrix form [see (4)],

$$S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{13}$$

Solving the Secular equation, we can find the eigenvalues and eigenvectors as

$$\lambda_1 = \frac{\hbar}{2} \tag{14}$$

with the corresponding eigenvector

$$|y_1\rangle = \frac{1}{\sqrt{2}}(-i|+\rangle + |-\rangle) \tag{15}$$

and

$$\lambda_2 = -\frac{\hbar}{2} \tag{16}$$

with the corresponding eigenvector

$$|y_2\rangle = \frac{1}{\sqrt{2}}(i|+\rangle + |-\rangle) \tag{17}$$

(4) From (1), it is easy to write down

$$S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{18}$$

Then, compute the matrix elements of S_x as

$$\langle +|S_x|+\rangle \;=\; \langle +|\frac{\hbar}{2}[|+\rangle\langle -| + |-\rangle\langle +|]|+\rangle = 0 \tag{19}$$

$$\langle +|S_x|-\rangle = \langle +|\frac{\hbar}{2}[|+\rangle\langle -| + |-\rangle\langle +|]|-\rangle = \frac{\hbar}{2} \tag{20}$$

$$\langle -|S_x|+\rangle = \langle -|\frac{\hbar}{2}[|+\rangle\langle -| + |-\rangle\langle +|]|+\rangle = \frac{\hbar}{2} \tag{21}$$

$$\langle -|S_x|-\rangle = \langle -|\frac{\hbar}{2}[|+\rangle\langle -| + |-\rangle\langle +|]|-\rangle = 0 \tag{22}$$

Therefore the matrix of S_x is

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{23}$$

Nest, compute the matrix elements of S_y as

$$\langle +|S_y|+\rangle = \langle +|\frac{\hbar}{2}[i|-\rangle\langle +| - i|+\rangle\langle -|]|+\rangle = 0 \tag{24}$$

$$\langle +|S_y|-\rangle = \langle +|\frac{\hbar}{2}[i|-\rangle\langle +| - i|+\rangle\langle -|]|-\rangle = -i\frac{\hbar}{2} \tag{25}$$

$$\langle -|S_y|+\rangle = \langle -|\frac{\hbar}{2}[i|-\rangle\langle +| - i|+\rangle\langle -|]|+\rangle = i\frac{\hbar}{2} \tag{26}$$

$$\langle -|S_y|-\rangle = \langle -|\frac{\hbar}{2}[i|-\rangle\langle +| - i|+\rangle\langle -|]|-\rangle = 0 \tag{27}$$

Therefore the matrix of S_x is

$$S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{28}$$

(5) From (3), we know the eigenstates of S_y are

$$|y_1\rangle = \frac{1}{\sqrt{2}}(-i|+\rangle + |-\rangle) \tag{29}$$

$$|y_2\rangle = \frac{1}{\sqrt{2}}(i|+\rangle + |-\rangle) \tag{30}$$

Then the matrix elements of S_x in this set of basis is

$$\begin{aligned} S_x(1,1) &= \langle y_1|S_x|y_1\rangle \\ &= \frac{1}{2}(i\langle +| + \langle -|)\frac{\hbar}{2}[|+\rangle\langle -| + |-\rangle\langle +|](-i|+\rangle + |-\rangle) \\ &= 0 \end{aligned} \tag{31}$$

$$\begin{aligned} S_x(1,2) &= \langle y_1|S_x|y_2\rangle \\ &= \frac{1}{2}(i\langle +| + \langle -|)\frac{\hbar}{2}[|+\rangle\langle -| + |-\rangle\langle +|](i|+\rangle + |-\rangle) \\ &= i\frac{\hbar}{2} \end{aligned} \tag{32}$$

$$\begin{aligned} S_x(2,1) &= \langle y_2|S_x|y_1\rangle \\ &= \frac{1}{2}(-i\langle +| + \langle -|)\frac{\hbar}{2}[|+\rangle\langle -| + |-\rangle\langle +|](-i|+\rangle + |-\rangle) \\ &= -i\frac{\hbar}{2} \end{aligned} \tag{33}$$

$$\begin{aligned} S_x(2,2) &= \langle y_2|S_x|y_2\rangle \\ &= \frac{1}{2}(-i\langle +| + \langle -|)\frac{\hbar}{2}[|+\rangle\langle -| + |-\rangle\langle +|](i|+\rangle + |-\rangle) \\ &= 0 \end{aligned} \tag{34}$$

So, the matrix form of S_x is

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \tag{35}$$

The matrix elements of S_z in this set of basis is

$$\begin{aligned} S_z(1,1) &= \langle y_1|S_z|y_1\rangle \\ &= \frac{1}{2}(i\langle +| + \langle -|)\frac{\hbar}{2}[|+\rangle\langle +| - |-\rangle\langle -|](-i|+\rangle + |-\rangle) \\ &= 0 \end{aligned} \tag{36}$$

$$\begin{aligned} S_z(1,2) &= \langle y_1|S_z|y_2\rangle \\ &= \frac{1}{2}(i\langle +| + \langle -|)\frac{\hbar}{2}[|+\rangle\langle +| - |-\rangle\langle -|](i|+\rangle + |-\rangle) \\ &= -\frac{\hbar}{2} \end{aligned} \tag{37}$$

$$\begin{aligned} S_z(2,1) &= \langle y_2|S_z|y_1\rangle \\ &= \frac{1}{2}(-i\langle +| + \langle -|)\frac{\hbar}{2}[|+\rangle\langle +| - |-\rangle\langle -|](-i|+\rangle + |-\rangle) \\ &= -\frac{\hbar}{2} \end{aligned} \tag{38}$$

$$\begin{aligned} S_z(2,2) &= \langle y_2|S_z|y_2\rangle \\ &= \frac{1}{2}(-i\langle +| + \langle -|)\frac{\hbar}{2}[|+\rangle\langle +| - |-\rangle\langle -|](i|+\rangle + |-\rangle) \\ &= 0 \end{aligned} \tag{39}$$

So, the matrix form of S_z is

$$S_z = \frac{\hbar}{2}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{40}$$

(6) From (3), the eigenvalues of S_x are $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$. So, the measurement outcome must be $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.

(7) According to the measurement theory, the probability of obtaining $\frac{\hbar}{2}$ in the measurement on $S_x$is

$$\begin{aligned} P_1 &= |\langle x_1|\psi\rangle|^2 \\ &= |\frac{1}{\sqrt{2}}(\langle +| + \langle -|)(\cos\alpha|+\rangle + \sin\alpha|-\rangle)|^2 \\ &= |\frac{1}{\sqrt{2}}(\cos\alpha + \sin\alpha)|^2 \\ &= \frac{1}{2}|\cos\alpha + \sin\alpha|^2 \end{aligned}$$

the probability of obtaining $-\frac{\hbar}{2}$ in the measurement on S_x is

$$\begin{aligned} P_2 &= |\langle x_2|\psi\rangle|^2 \\ &= |\frac{1}{\sqrt{2}}(\langle +| - \langle -|)(\cos\alpha|+\rangle + \sin\alpha|-\rangle)|^2 \\ &= |\frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha)|^2 \\ &= \frac{1}{2}|\cos\alpha - \sin\alpha|^2 \end{aligned}$$

(8) It is impossible to determine S_x and S_y at the same time, because they are incompatible operators (do not commute). This reflects the uncertainty relation which is a fundamental relation of quantum mechanics.

II. PART II

(1) First, we write down the matrix form of $\sigma_x \otimes \sigma_x$ and $\sigma_x \otimes \sigma_y$

$$\sigma_x \otimes \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{41}$$

$$\sigma_x \otimes \sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \tag{42}$$

then, it is clear to check

$$(\sigma_x \otimes \sigma_x)^\dagger = \sigma_x \otimes \sigma_x \tag{43}$$

$$(\sigma_y \otimes \sigma_y)^\dagger = \sigma_y \otimes \sigma_y \tag{44}$$

which means they are really hermitian matrix.

(2) Solving the Secular equation, we find the eigenvalues and eigenvectors are

$$\lambda_1 = 1 \tag{45}$$

with the corresponding eigenvector

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{46}$$

$$\lambda_2 = 1 \tag{47}$$

with the corresponding eigenvector

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \tag{48}$$

$$\lambda_3 = -1 \tag{49}$$

with the corresponding eigenvector

$$|\psi_3\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{50}$$

$$\lambda_4 = -1 \tag{51}$$

with the corresponding eigenvector

$$|\psi_4\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \tag{52}$$

(3) According to the measurement theory, the measurement results are the eigenvalues λ_i, i.e., 1 or -1.

(4) Write down the given state in matrix form as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \tag{53}$$

This is just the second eigenstate $|\psi_2\rangle$. So, the measurement outcome will be 1, and we will find this state with probability 1, and the probability of finding other states are zero.

We can check this by computing the probabilities of finding each eigenstates as

$$P_1 = |\langle\psi_1|\psi\rangle|^2 = 0 \tag{54}$$

$$P_2 = |\langle\psi_2|\psi\rangle|^2 = 1 \tag{55}$$

$$P_3 = |\langle\psi_3|\psi\rangle|^2 = 0 \tag{56}$$

$$P_4 = |\langle\psi_3|\psi\rangle|^2 = 0 \tag{57}$$

This also prove we will find the second eigenstate with probability 1.

HW5a solution

1. See Sakurai, page 35-36 Eq. (1.4.53-1.4.63)
2. (a) In classical mechanics,

$$[x, F(p_x)] = \frac{\partial x}{\partial x}\frac{\partial F}{\partial p_x} - \frac{\partial x}{\partial p_x}\frac{\partial F}{\partial x} = \frac{\partial F(p_x)}{\partial p_x}. \tag{1}$$

(b) In quantum mechanics,

$$\begin{aligned}
[x, \exp(\frac{ip_x a}{\hbar})] &= [x, \sum_n \frac{(\frac{ip_x a}{\hbar})^n}{n!}] \\
&= \sum_n [x, \frac{(\frac{ip_x a}{\hbar})^n}{n!}] \\
&= \sum_n \frac{(\frac{ia}{\hbar})^n}{n!}(in\hbar p^{n-1}) \\
&= -a\sum_n \frac{(\frac{ip_x a}{\hbar})^n}{n!} \\
&= -a\exp(\frac{ip_x a}{\hbar}).
\end{aligned} \tag{2}$$

(c) Let $|\psi\rangle = \exp(\frac{ip_x a}{\hbar})|x'\rangle$, then

$$\begin{aligned}
x|\psi\rangle &= x\exp(\frac{ip_x a}{\hbar})|x'\rangle \\
&= \{\exp(\frac{ip_x a}{\hbar})| + [x, \exp(\frac{ip_x a}{\hbar})|]\}|x'\rangle. \\
&= (x' - a)|\psi\rangle
\end{aligned} \tag{3}$$

So, $|\psi\rangle$ is an eigenstate of x, with the corresponding eigenvalue $(x' - a)$.

3. Since we know

$$\langle x|p'\rangle = \psi_{p'(x)} = \frac{1}{\sqrt{2\pi\hbar}}e^{ip'x/\hbar}, \tag{4}$$

which means the state $|p'\rangle$ can be expanded in the $|x\rangle$ basis as

$$|p'\rangle = \int dx \frac{1}{\sqrt{2\pi\hbar}}e^{ip'x/\hbar}|x\rangle. \tag{5}$$

Therefore,

$$\begin{aligned}
\langle p'|\hat{x}|\alpha\rangle &= \int dx \frac{1}{\sqrt{2\pi\hbar}}e^{-ip'x/\hbar}\langle x|\hat{x}|\alpha\rangle \\
&= \int dx \frac{1}{\sqrt{2\pi\hbar}}e^{-ip'x/\hbar}x\langle x|\alpha\rangle.
\end{aligned} \tag{6}$$

On the other hand,

$$\begin{aligned} i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle &= i\hbar\frac{\partial}{\partial p'}\int dx\frac{1}{\sqrt{2\pi\hbar}}e^{-ip'x/\hbar}\langle x|\alpha\rangle \\ &= i\hbar\int dx\frac{1}{\sqrt{2\pi\hbar}}\frac{\partial}{\partial p'}e^{-ip'x/\hbar}\langle x|\alpha\rangle \\ &= i\hbar\int dx\frac{1}{\sqrt{2\pi\hbar}}(-ix/\hbar)e^{-ip'x/\hbar}\langle x|\alpha\rangle \\ &= \int dx\frac{1}{\sqrt{2\pi\hbar}}e^{-ip'x/\hbar}x\langle x|\alpha\rangle. \end{aligned} \tag{7}$$

Finally, we prove

$$\langle p'|\hat{x}|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle \tag{8}$$

Then, we can write down the Hamiltonian in the momentum basis as

$$H = \frac{p^2}{2m} - \frac{m\omega^2\hbar^2}{2}\frac{\partial^2}{\partial p^2}, \tag{9}$$

by taking

$$\hat{x} \to i\hbar\frac{\partial}{\partial p}, \quad \hat{p} \to p. \tag{10}$$

HW5b solution

1. Sakruai 2.13 part (a) only

The operators x and p can be expressed as

$$x = \sqrt{\hbar/2m\omega}(a + a^\dagger), \quad p = i\sqrt{\hbar m\omega/2}(a^\dagger - a) \tag{1}$$

Then, we have

$$\begin{aligned}
\langle m|x|n\rangle &= \langle m|\sqrt{\hbar/2m\omega}(a + a^\dagger)|n\rangle \\
&= \sqrt{\hbar/2m\omega}(\langle m|\sqrt{n}|n-1\rangle + \langle m|\sqrt{n+1}|n+1\rangle) \\
&= \sqrt{\hbar/2m\omega}(\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1})
\end{aligned} \tag{2}$$

$$\begin{aligned}
\langle m|p|n\rangle &= \langle m|i\sqrt{\hbar m\omega/2}(a^\dagger - a)|n\rangle \\
&= i\sqrt{\hbar m\omega/2}(\langle m|\sqrt{n+1}|n+1\rangle - \langle m|\sqrt{n}|n-1\rangle) \\
&= \sqrt{\hbar/2m\omega}(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1})
\end{aligned} \tag{3}$$

$$\begin{aligned}
\langle m|\{x,p\}|n\rangle &= \langle m|(xp + px)|n\rangle \\
&= \frac{i\hbar}{2}\langle m|(aa^\dagger - aa + a^\dagger a^\dagger - a^\dagger a)|n\rangle \\
&\quad + \frac{i\hbar}{2}\langle m|(a^\dagger a + a^\dagger a^\dagger - aa - aa^\dagger)|n\rangle \\
&= \frac{i\hbar}{2}\langle m|(1 - aa + a^\dagger a^\dagger)|n\rangle \\
&\quad + \frac{i\hbar}{2}\langle m|(a^\dagger a^\dagger - aa - 1)|n\rangle \\
&= i\hbar(\sqrt{(n+1)(n+2)}\delta_{m,n+2} - \sqrt{n(n-1)}\delta_{m,n-2})
\end{aligned} \tag{4}$$

$$\begin{aligned}
\langle m|x^2|n\rangle &= \frac{\hbar}{2m\omega}\langle m|(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger)|n\rangle \\
&= \frac{\hbar}{2m\omega}[\sqrt{n(n-1)}\delta_{m,n-2} + \sqrt{(n+1)(n+2)}\delta_{m,n+2} + (2n+1)\delta_{m,n}]
\end{aligned} \tag{5}$$

$$\begin{aligned}
\langle m|p^2|n\rangle &= -\frac{m\omega\hbar}{2}\langle m|(a^\dagger a^\dagger - a^\dagger a - aa^\dagger + aa)|n\rangle \\
&= -\frac{\hbar}{2m\omega}[\sqrt{n(n-1)}\delta_{m,n-2} + \sqrt{(n+1)(n+2)}\delta_{m,n+2} - (2n+1)\delta_{m,n}]
\end{aligned} \tag{6}$$

2. Proof:

(a) Using the definition

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{7}$$

$$\begin{aligned}
\langle\alpha|\alpha\rangle &= e^{-|\alpha|^2}\sum_{m,n}^{\infty}\langle m|\frac{(\alpha^*)^m\alpha^n}{\sqrt{m!n!}}|n\rangle \\
&= e^{-|\alpha|^2}\sum_{m,n}^{\infty}\frac{(\alpha^*)^m\alpha^n}{\sqrt{m!n!}}\langle m|n\rangle \\
&= e^{-|\alpha|^2}\sum_{m,n}^{\infty}\frac{(\alpha^*)^m\alpha^n}{\sqrt{m!n!}}\delta_{m,n} \\
&= e^{-|\alpha|^2}e^{|\alpha|^2} \\
&= 1
\end{aligned} \tag{8}$$

So, the coherent state is normalized.

(b) Similar to Problem 1, the operators x and p can be expressed as

$$x = \sqrt{\hbar/2m\omega}(a + a^\dagger), \quad p = i\sqrt{\hbar m\omega/2}(a^\dagger - a) \tag{9}$$

Then,

$$\begin{aligned}
\langle\alpha|x|\alpha\rangle &= \langle\alpha|\sqrt{\hbar/2m\omega}(a + a^\dagger)|\alpha\rangle \\
&= \sqrt{\hbar/2m\omega}(\alpha + \alpha^*)
\end{aligned} \tag{10}$$

$$\begin{aligned}
\langle\alpha|p|\alpha\rangle &= \langle\alpha|i\sqrt{\hbar m\omega/2}(a^\dagger - a)|\alpha\rangle \\
&= i\sqrt{\hbar m\omega/2}(\alpha^* - \alpha)
\end{aligned} \tag{11}$$

$$\begin{aligned}
\langle\alpha|x^2|\alpha\rangle &= \frac{\hbar}{2m\omega}\langle\alpha|(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger)|\alpha\rangle \\
&= \frac{\hbar}{2m\omega}[\alpha^2 + 1 + 2|\alpha|^2 + (\alpha^*)^2]
\end{aligned} \tag{12}$$

$$\begin{aligned}
\langle\alpha|p^2|\alpha\rangle &= -\frac{m\omega\hbar}{2}\langle\alpha|(a^\dagger a^\dagger - a^\dagger a - aa^\dagger + aa)|\alpha\rangle \\
&= -\frac{m\omega\hbar}{2}[\alpha^2 - 1 - 2|\alpha|^2 + (\alpha^*)^2]
\end{aligned} \tag{13}$$

$$\begin{aligned}
\langle\alpha|(\Delta x)^2|\alpha\rangle &= \langle\alpha|x^2|\alpha\rangle - \langle\alpha|x|\alpha\rangle^2 \\
&= \frac{\hbar}{2m\omega}[\alpha^2 + 1 + 2|\alpha|^2 + (\alpha^*)^2] - \frac{\hbar}{2m\omega}(\alpha + \alpha^*)^2 \\
&= \frac{\hbar}{2m\omega}
\end{aligned} \tag{14}$$

$$
\begin{aligned}
\langle\alpha|(\Delta p)^2|\alpha\rangle &= \langle\alpha|p^2|\alpha\rangle - \langle\alpha|p|\alpha\rangle^2 \\
&= -\frac{m\omega\hbar}{2}[\alpha^2 - 1 - 2|\alpha|^2 + (\alpha^*)^2] + \frac{\hbar}{2m\omega}(\alpha^* - \alpha)^2 \\
&= \frac{m\omega\hbar}{2} \qquad (15)
\end{aligned}
$$

Finally,

$$
\langle\alpha|(\Delta x)^2|\alpha\rangle\langle\alpha|(\Delta p)^2|\alpha\rangle = \frac{\hbar}{2m\omega}\frac{m\omega\hbar}{2} = \frac{\hbar^2}{4} \qquad (16)
$$

So, coherent state gives the minimum uncertainty relation.

(c) Suppose the so-called "N-P" state exist, it must be a linear combination of Fock states as

$$
|\beta\rangle = \sum_{n=0}^{\infty} c_n|n\rangle, \qquad (17)
$$

then,

$$
a^\dagger|\beta\rangle = a^\dagger \sum_{n=0}^{\infty} c_n|n\rangle = \sum_{n=0}^{\infty} c_n\sqrt{n+1}|n+1\rangle. \qquad (18)
$$

From this equation, it is easy to see $a^\dagger|\beta\rangle$ do not contain $|0\rangle$ component, on the other hand, $a^\dagger|\beta\rangle = \beta|\beta\rangle$, so $\beta|\beta\rangle = \sum_{n=0}^{\infty} c_n\beta|n\rangle$ also contain no $|0\rangle$ component, i.e., $c_0 = 0$.

Next, from the fact $|\beta\rangle$is the eigenstate of $a^\dagger$

$$
a^\dagger|\beta\rangle = \beta|\beta\rangle \qquad (19)
$$

we have

$$
\sum_{n=0}^{\infty} c_n\sqrt{n+1}|n+1\rangle = \sum_{n=0}^{\infty} c_n\beta|n\rangle \qquad (20)
$$

Therefore, we derive the recursion formula for the coefficients

$$
c_{n+1}\beta = c_n\sqrt{n+1} \qquad (21)
$$

From this equation, it is clearly that if $c_0 = 0$, then all the coefficients c_n are all zero. Finally, the state $|\beta\rangle$ become a trivial state

$$
|\beta\rangle = 0 \qquad (22)
$$

which is not a well-defined state (its norm is zero, not 1).

HW6 solution

1. (1) What we need to prove is

$$i\hbar\frac{\partial}{\partial t}U(t) = HU(t) \tag{1}$$

Simply substituting the definition of $U(t)$

$$U(t) = e^{-\frac{i}{\hbar}Ht} \tag{2}$$

into the Schrödinger equation, we have

$$i\hbar\frac{\partial}{\partial t}U(t) = i\hbar\frac{\partial}{\partial t}[e^{-\frac{i}{\hbar}Ht}] = i\hbar(-\frac{i}{\hbar}H)e^{-\frac{i}{\hbar}Ht} = HU(t) \tag{3}$$

which implies $U(t)$ satisfies the Schrödinger equation. $(t \rightarrow (t-t_0))$

(2)

(a)

$$U(t,t_0) = e^{-\frac{i}{\hbar}H(t-t_0)} = e^{\frac{i}{\hbar}\frac{eB\hbar}{2mc}\sigma_z(t-t_0)} \tag{4}$$

Taking the Taylor expansion as

$$\begin{aligned}
e^{\frac{i}{\hbar}\frac{eB\hbar}{2mc}\sigma_z(t-t_0)} &= \sum_n \frac{1}{n!}[\frac{eB\hbar}{2mc}(t-t_0)]^n(\sigma_z)^n \\
&= \sum_n \frac{1}{n!}[\frac{eB\hbar}{2mc}(t-t_0)]^n(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix})^n \\
&= \sum_n \frac{1}{n!}[\frac{eB\hbar}{2mc}(t-t_0)]^n \begin{bmatrix} 1^n & 0 \\ 0 & (-1)^n \end{bmatrix}) \\
&= \begin{bmatrix} \exp[\frac{eB\hbar}{2mc}(t-t_0)] & 0 \\ 0 & \exp[-\frac{eB\hbar}{2mc}(t-t_0)] \end{bmatrix}
\end{aligned}$$

Finally, in the matrix form, it is

$$U(t) = \begin{bmatrix} \exp[\frac{eB\hbar}{2mc}(t-t_0)] & 0 \\ 0 & \exp[-\frac{eB\hbar}{2mc}(t-t_0)] \end{bmatrix} \tag{5}$$

(b)

$$\begin{aligned}|\alpha,t\rangle &= U|\alpha\rangle \\ &= \begin{bmatrix} e^{\frac{i}{\hbar}\frac{eB\hbar}{2mc}(t-t_0)} & 0 \\ 0 & e^{-\frac{i}{\hbar}\frac{eB\hbar}{2mc}(t-t_0)} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} ae^{\frac{i}{\hbar}\frac{eB\hbar}{2mc}(t-t_0)} & 0 \\ 0 & be^{-\frac{i}{\hbar}\frac{eB\hbar}{2mc}(t-t_0)} \end{bmatrix} \end{aligned} \tag{6}$$

2. (1) What we need to prove is

$$i\hbar\frac{\partial}{\partial t}U(t) = H(t)U(t) \tag{7}$$

Simply substituting the definition of $U(t)$

$$U(t) = e^{-\frac{i}{\hbar}\int_{t0}^{t} H(t')dt'} \tag{8}$$

into the Schrödinger equation, we have

$$i\hbar\frac{\partial}{\partial t}U(t) = i\hbar\frac{\partial}{\partial t}[e^{-\frac{i}{\hbar}\int_{t0}^{t} H(t')dt'}] = i\hbar[-\frac{i}{\hbar}H(t)]U(t) = H(t)U(t) \tag{9}$$

which implies $U(t)$ satisfies the Schrödinger equation. ($\frac{d}{dt}\int_{t0}^{t} H(t')dt' = H(t-t_0)$)

(2)

(a)

$$U(t,t_0) = e^{-\frac{i}{\hbar}\int_{t0}^{t} H(t')dt'} = e^{\frac{i}{\hbar}\int_{t0}^{t} \frac{eB(t')\hbar}{2mc}dt'\sigma_z} \tag{10}$$

In the matrix form, it is

$$U(t) = \begin{bmatrix} e^{\frac{i}{\hbar}\int_{t0}^{t}\frac{eB(t')\hbar}{2mc}dt'} & 0 \\ 0 & e^{-\frac{i}{\hbar}\int_{t0}^{t}\frac{eB(t')\hbar}{2mc}dt'} \end{bmatrix} \tag{11}$$

(b)

$$\begin{aligned}|\alpha,t\rangle &= U|\alpha\rangle \\ &= \begin{bmatrix} e^{\frac{i}{\hbar}\int_{t0}^{t}\frac{eB(t')\hbar}{2mc}dt'} & 0 \\ 0 & e^{-\frac{i}{\hbar}\int_{t0}^{t}\frac{eB(t')\hbar}{2mc}dt'} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} ae^{\frac{i}{\hbar}\int_{t0}^{t}\frac{eB(t')\hbar}{2mc}dt'} & 0 \\ 0 & be^{-\frac{i}{\hbar}\int_{t0}^{t}\frac{eB(t')\hbar}{2mc}dt'} \end{bmatrix} \end{aligned} \tag{12}$$

3. (a) In the basis $\{|L\rangle, |R\rangle\}$, the Hamiltonian can be written as

$$H = \Delta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{13}$$

Solve the eigen-equation $H|\psi\rangle = E|\psi\rangle$, we will derive the eigenvalues and eigenvectors as

(1) $E_1 = \Delta$ with the corresponding eigenvector

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{14}$$

(2) $E_2 = -\Delta$ with the corresponding eigenvector

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{15}$$

(b) The time evolution operator is

$$U(t) = \exp(-\frac{i}{\hbar}Ht) = \exp[-\frac{i}{\hbar}\Delta(|L\rangle\langle R| + |R\rangle\langle L|)t] \tag{16}$$

and

$$|L\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$$
$$|R\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle - |\psi_2\rangle)$$

The initial state is

$$|\psi(0)\rangle = \langle L|\alpha\rangle|L\rangle + \langle R|\alpha\rangle|R\rangle \tag{17}$$

Applying this operator to the initial state we will find the final state as

$$\begin{aligned} U(t)|\psi(0)\rangle &= \exp[-\frac{i}{\hbar}\Delta Ht](\langle L|\alpha\rangle|L\rangle + \langle R|\alpha\rangle|R\rangle) \\ &= \exp[-\frac{i}{\hbar}\Delta Ht]\frac{1}{\sqrt{2}}[\langle L|\alpha\rangle(|\psi_1\rangle + |\psi_2\rangle) + \langle R|\alpha\rangle(|\psi_1\rangle - |\psi_2\rangle)] \\ &= \frac{1}{\sqrt{2}}\langle L|\alpha\rangle(e^{-\frac{i}{\hbar}\Delta t}|\psi_1\rangle + e^{\frac{i}{\hbar}\Delta t}|\psi_2\rangle) \\ &+ \frac{1}{\sqrt{2}}\langle R|\alpha\rangle(e^{-\frac{i}{\hbar}\Delta t}|\psi_1\rangle - e^{\frac{i}{\hbar}\Delta t}|\psi_2\rangle) \end{aligned} \tag{18}$$

where we use

$$H|\psi_1\rangle = \Delta|\psi_1\rangle \tag{19}$$

$$H|\psi_2\rangle = -\Delta|\psi_2\rangle \tag{20}$$

In the basis $\{|L\rangle, |R\rangle\}$, the final state is just

$$\begin{aligned} |\alpha, t\rangle &= U(t)|\psi(0)\rangle \\ &= (\langle L|\alpha\rangle \cos\omega t - i\langle R|\alpha\rangle \sin\omega t)|L\rangle \\ &+ (\langle R|\alpha\rangle \cos\omega t - i\langle L|\alpha\rangle \sin\omega t)|R\rangle \end{aligned} \tag{21}$$

where $\omega = \frac{\Delta}{\hbar}$.

(c) If the initial condition is that the particle in the right side, i.e.,

$$|\psi(0)\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{22}$$

Finally, the solution is

$$|\psi(t)\rangle = \begin{bmatrix} \sin\omega t \\ \cos\omega t \end{bmatrix} \tag{23}$$

and the probability is just

$$\langle L|\psi(t)\rangle = \sin^2\omega t \tag{24}$$

(d) In the Schrödinger picture, the Schrödinger equation can be written as

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix} A(t) \\ B(t) \end{bmatrix} = \Delta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A(t) \\ B(t) \end{bmatrix} \tag{25}$$

The coupled equation is

$$i\hbar\frac{\partial}{\partial t}A(t) = \Delta B(t) \tag{26}$$

$$i\hbar\frac{\partial}{\partial t}B(t) = \Delta A(t) \tag{27}$$

Solving this equation, we will find the solution is

$$A(t) = A_1\cos\omega t + A_2\sin\omega t \tag{28}$$

$$B(t) = B_1\cos\omega t + B_2\sin\omega t \tag{29}$$

Using the initial condition that

$$A(0) = \langle L|\alpha\rangle \tag{30}$$

$$B(0) = \langle R|\alpha\rangle \tag{31}$$

we will find the final solution is just the same as it is in (b) which is derived by using the evolution operator.

(e) It is easy to find the Hamiltonian is not Hermitian any more, then the evolution will not be unitary evolution any more by stone theorem. Therefore the probability may not be preserved. Here, we will not show the result explicit. In fact, it is very easy to check this conclusion.

4. If $t_1 < t_2$, the right hand side (RHS) is

$$I_1 = \frac{1}{2}\int_0^t dt_2 \int_0^{t_2} dt_1 \hat{H}(t_2)\hat{H}(t_1). \tag{32}$$

If $t_1 > t_2$, the right hand side is

$$I_2 = \frac{1}{2}\int_0^t dt_1 \int_0^{t_1} dt_2 \hat{H}(t_1)\hat{H}(t_2). \tag{33}$$

Note that the indexes t_1 and t_2 are dummy integral indexes, interchanging them we will found $I_1 = I_2$. Therefore, RHS is

$$RHS = I_1 + I_2 = 2I_2 = \int_0^t dt_1 \int_0^{t_1} dt_2 \hat{H}(t_1)\hat{H}(t_2) = LHS. \tag{34}$$

The proof above can be easily understood by looking at the integral regions plotted in Fig. 1.

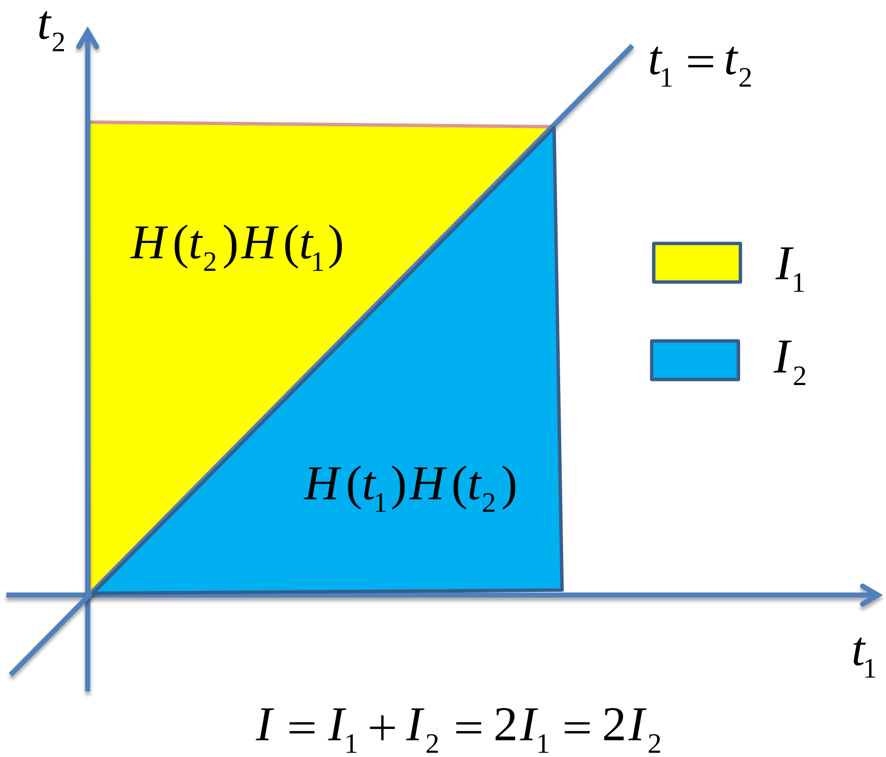

Figure 1: Integral region

HW7 solution

1. (1) Simply substituting

$$a = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{ip}{m\omega}) \tag{1}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(x - \frac{ip}{m\omega}) \tag{2}$$

into

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{3}$$

we obtain

$$\begin{aligned} H &= \hbar\omega\left[\sqrt{\frac{m\omega}{2\hbar}}(x - \frac{ip}{m\omega})\sqrt{\frac{m\omega}{2\hbar}}(x + \frac{ip}{m\omega}) + \frac{1}{2}\right] \\ &= \frac{m\omega^2}{2}(x^2 + \frac{ixp}{m\omega} - \frac{ipx}{m\omega} + \frac{p^2}{m^2\omega^2}) + \frac{1}{2}\hbar\omega \\ &= \frac{m\omega^2}{2}(x^2 - \frac{\hbar}{m\omega} + \frac{p^2}{m^2\omega^2}) + \frac{1}{2}\hbar\omega \\ &= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \end{aligned} \tag{4}$$

Thus, we prove the two Hamiltonian are identical.

(2) In the Heisenberg picture,

$$\begin{aligned} a(t) &= e^{iHt/\hbar} a e^{-iHt/\hbar} \\ &= a + \frac{it}{\hbar}[H, a] + \frac{(it)^2}{2!\hbar^2}[H, [H, a]] + \cdots \\ &= a + \frac{it}{\hbar}(-\hbar\omega)a + \frac{(it)^2}{2!\hbar^2}(-\hbar\omega)^2 a + \cdots \\ &= a \exp(-i\omega t) \end{aligned} \tag{5}$$

$$\begin{aligned} a^\dagger(t) &= e^{iHt/\hbar} a^\dagger e^{-iHt/\hbar} \\ &= a^\dagger + \frac{it}{\hbar}[H, a^\dagger] + \frac{(it)^2}{2!\hbar^2}[H, [H, a^\dagger]] + \cdots \\ &= a^\dagger + \frac{it}{\hbar}(\hbar\omega)a^\dagger + \frac{(it)^2}{2!\hbar^2}(\hbar\omega)^2 a^\dagger + \cdots \\ &= a^\dagger \exp(i\omega t) \end{aligned} \tag{6}$$

(3) The Heisenberg equations are

$$\begin{aligned}\frac{d}{dt}a(t) &= \frac{i}{\hbar}[H,a] \\ &= \frac{i}{\hbar}[\hbar\omega(a^\dagger a+\frac{1}{2}),a] \\ &= -i\omega a\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{d}{dt}a^\dagger(t) &= \frac{i}{\hbar}[H,a^\dagger] \\ &= \frac{i}{\hbar}[\hbar\omega(a^\dagger a+\frac{1}{2}),a^\dagger] \\ &= i\omega a^\dagger\end{aligned} \tag{8}$$

(4) According to the equations above associated with the boundary conditions

$$a(0) = a \tag{9}$$

$$a^\dagger(0) = a^\dagger \tag{10}$$

the solution is

$$a(t) = a\exp(-i\omega t) \tag{11}$$

$$a^\dagger(t) = a^\dagger\exp(i\omega t) \tag{12}$$

2. (a) Arbitrary linear combination of $|0\rangle$and $|1\rangle$can be expressed as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{13}$$

then, the mean value of $x = \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)$ is

$$\begin{aligned}\langle\psi|x|\psi\rangle &= (\alpha^*\langle 0| + \beta^*\langle 1|)\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)(\alpha|0\rangle+\beta|1\rangle) \\ &= \sqrt{\frac{\hbar}{2m\omega}}(\alpha^*\langle 0|+\beta^*\langle 1|)(\beta|0\rangle+\alpha|1\rangle+\sqrt{2}\beta|2\rangle) \\ &= \sqrt{\frac{\hbar}{2m\omega}}(\alpha^*\beta+\beta^*\alpha)\end{aligned}$$

when $\alpha = \beta$ the mean value take the largest value, e.g.,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{14}$$

give the largest mean value.

(b) In the Schrödinger picture, the state will evolve as

$$\begin{aligned}\frac{d}{dt}|\psi(t)\rangle &= -\frac{i}{\hbar}\hbar\omega(a^\dagger a+\frac{1}{2})|\psi(t)\rangle \\ &= -i\omega(a^\dagger a+\frac{1}{2})|\psi(t)\rangle\end{aligned}$$

Using the boundary condition

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \tag{15}$$

and notice the result in (a), the solution is

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(e^{-\frac{1}{2}i\omega t}|0\rangle+e^{-\frac{3}{2}i\omega t}|1\rangle) \tag{16}$$

$$\begin{aligned}\langle\psi(t)|x|\psi(t)\rangle &= \sqrt{\frac{\hbar}{2m\omega}}(\frac{1}{2}e^{\frac{1}{2}i\omega t}e^{-\frac{3}{2}i\omega t}+\frac{1}{2}e^{-\frac{1}{2}i\omega t}e^{\frac{3}{2}i\omega t}) \\ &= \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)\end{aligned} \tag{17}$$

In the Heisenberg picture,

$$a(t) = a\exp(-i\omega t) \tag{18}$$

$$a^\dagger(t) = a^\dagger\exp(i\omega t) \tag{19}$$

The mean value at time t is

$$\begin{aligned}\langle\psi(0)|x(t)|\psi(0)\rangle &= \frac{1}{\sqrt{2}}(\langle 0|+\langle 1|)\sqrt{\frac{\hbar}{2m\omega}}[a(t)+a^\dagger(t)]\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \\ &= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}(\langle 0|+\langle 1|)(e^{-i\omega t}|0\rangle+e^{i\omega t}|1\rangle+e^{i\omega t}|2\rangle) \\ &= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}(e^{-i\omega t}+e^{i\omega t}) \\ &= \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)\end{aligned} \tag{20}$$

which is the same as we derived in Schrödinger picture.

(c) We have already compute $\langle x\rangle$ in (b), here, we just need to compute $\langle x^2\rangle = \frac{\hbar}{2m\omega}\langle aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger\rangle$. In the Schrödinger picture, the state will evolve as

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(e^{-\frac{1}{2}i\omega t}|0\rangle+e^{-\frac{3}{2}i\omega t}|1\rangle) \tag{21}$$

$$\begin{aligned}
\langle\psi(t)|x^2|\psi(t)\rangle &= \frac{\hbar}{4m\omega}(e^{\frac{1}{2}i\omega t}\langle 0| + e^{\frac{3}{2}i\omega t}\langle 1|)(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger)(e^{-\frac{1}{2}i\omega t}|0\rangle + e^{-\frac{3}{2}i\omega t}|1\rangle) \\
&= \frac{\hbar}{4m\omega}(e^{\frac{1}{2}i\omega t}\langle 0| + e^{\frac{3}{2}i\omega t}\langle 1|)(e^{-\frac{1}{2}i\omega t}|0\rangle + 3e^{-\frac{3}{2}i\omega t}|1\rangle + \cdots) \\
&= \frac{\hbar}{4m\omega}(1+3) \\
&= \frac{\hbar}{m\omega} \qquad (22)
\end{aligned}$$

Finally,

$$\begin{aligned}
\langle(\Delta x)^2\rangle &= \langle x^2\rangle - \langle x\rangle^2 \\
&= \frac{\hbar}{m\omega} - \frac{\hbar}{2m\omega}\cos^2(\omega t) \\
&= \frac{\hbar}{m\omega}[1 - \frac{1}{2}\cos^2(\omega t)] \qquad (23)
\end{aligned}$$

In the Heisenberg picture,

$$x^2(t) = \frac{\hbar}{2m\omega}[aae^{-2i\omega t} + aa^\dagger + a^\dagger a + a^\dagger a^\dagger e^{2i\omega t}] \qquad (24)$$

$$a(t) = a\exp(-i\omega t) \qquad (25)$$

$$a^\dagger(t) = a^\dagger \exp(i\omega t) \qquad (26)$$

The mean value at time t is

$$\begin{aligned}
\langle\psi(0)|x^2(t)|\psi(0)\rangle &= \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\frac{\hbar}{2m\omega}[aae^{-2i\omega t} + aa^\dagger + a^\dagger a + a^\dagger a^\dagger e^{2i\omega t}]\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
&= \frac{1}{2}\frac{\hbar}{2m\omega}(\langle 0| + \langle 1|)(|0\rangle + 3|1\rangle + \cdots) \\
&= \frac{1}{2}\frac{\hbar}{2m\omega}(1+3) \\
&= \frac{\hbar}{m\omega} \qquad (27)
\end{aligned}$$

Finally,

$$\begin{aligned}
\langle(\Delta x)^2\rangle &= \langle x^2\rangle - \langle x\rangle^2 \\
&= \frac{\hbar}{m\omega} - \frac{\hbar}{2m\omega}\cos^2(\omega t) \\
&= \frac{\hbar}{m\omega}[1 - \frac{1}{2}\cos^2(\omega t)] \qquad (28)
\end{aligned}$$

which is the same as we derived in Schrödinger picture.

3. The left hand side is

$$\begin{aligned}\langle 0|e^{ikx}|0\rangle &= \langle 0|\exp[ik\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)]|0\rangle \\ &= \langle 0|\exp[ik\sqrt{\frac{\hbar}{2m\omega}}a^\dagger]\exp[ik\sqrt{\frac{\hbar}{2m\omega}}a]\exp[-\frac{\hbar k^2}{4m\omega}]|0\rangle \\ &= \exp[-\frac{\hbar k^2}{4m\omega}] \end{aligned} \tag{29}$$

Then, we compute the right hand side as

$$\begin{aligned}\exp[-k^2\langle 0|x^2|0\rangle/2] &= \exp[-k^2\frac{\hbar}{2m\omega}/2] \\ &= \exp[-\frac{\hbar k^2}{4m\omega}] \end{aligned} \tag{30}$$

where we use

$$\begin{aligned}\langle 0|x^2|0\rangle &= \frac{\hbar}{2m\omega}\langle 0|(aa+aa^\dagger+a^\dagger a+a^\dagger a^\dagger)|0\rangle \\ &= \frac{\hbar}{2m\omega} \end{aligned} \tag{31}$$

Finally, we prove that left hand side and right hand side are equal, i.e.,

$$\langle 0|e^{ikx}|0\rangle = \exp[-k^2\langle 0|x^2|0\rangle/2] \tag{32}$$

HW8a solution

(1) The Heisenberg equation of motion is

$$\frac{d}{dt}A(t) = \frac{i}{\hbar}[H, A(t)] \tag{1}$$

so

$$\begin{aligned}
\frac{d}{dt}a &= \frac{i}{\hbar}[H, a] \\
&= \frac{i}{\hbar}[\frac{\hbar\omega_A}{2}\sigma_z + \hbar\omega a^\dagger a + \hbar g(\sigma_- a^\dagger + \sigma_+ a), a] \\
&= -i\omega a - ig\sigma_-
\end{aligned} \tag{2}$$

$$\begin{aligned}
\frac{d}{dt}\sigma_- &= \frac{i}{\hbar}[H, \sigma_-] \\
&= \frac{i}{\hbar}[\frac{\hbar\omega_A}{2}\sigma_z + \hbar\omega a^\dagger a + \hbar g(\sigma_- a^\dagger + \sigma_+ a), \sigma_-] \\
&= -i\omega_A\sigma_- + ig\sigma_z a
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\sigma_z &= \frac{i}{\hbar}[H, \sigma_z] \\
&= \frac{i}{\hbar}[\frac{\hbar\omega_A}{2}\sigma_z + \hbar\omega a^\dagger a + \hbar g(\sigma_- a^\dagger + \sigma_+ a), \sigma_z] \\
&= 2ig\sigma_- a^\dagger - 2ig\sigma_+ a
\end{aligned} \tag{3}$$

(2) Proof:

$$\begin{aligned}
[N, H] &= [a^\dagger a, \frac{\hbar\omega_A}{2}\sigma_z + \hbar\omega a^\dagger a + \hbar g(\sigma_- a^\dagger + \sigma_+ a)] \\
&\quad + [\sigma_+\sigma_-, \frac{\hbar\omega_A}{2}\sigma_z + \hbar\omega a^\dagger a + \hbar g(\sigma_- a^\dagger + \sigma_+ a)] \\
&= (\hbar g\sigma_- a^\dagger - \hbar g\sigma_+ a) + (-\hbar g\sigma_- a^\dagger + \hbar g\sigma_+ a) \\
&= 0
\end{aligned} \tag{4}$$

$$
\begin{aligned}
[C,H] &= [\frac{1}{2}\Delta\sigma_z, \frac{\hbar\omega_A}{2}\sigma_z + \hbar\omega a^\dagger a + \hbar g(\sigma_- a^\dagger + \sigma_+ a)] \\
&\quad +[g\sigma_+ a, \frac{\hbar\omega_A}{2}\sigma_z + \hbar\omega a^\dagger a + \hbar g(\sigma_- a^\dagger + \sigma_+ a)] \\
&\quad +[g\sigma_- a^\dagger, \frac{\hbar\omega_A}{2}\sigma_z + \hbar\omega a^\dagger a + \hbar g(\sigma_- a^\dagger + \sigma_+ a)] \\
&= (-\Delta\hbar g\sigma_- a^\dagger + \Delta\hbar g\sigma_+ a) \\
&\quad +(-g\hbar\omega_A\sigma_+ a + \hbar g\omega\sigma_+ a) + \hbar g^2[\sigma_+ a, \sigma_- a] \\
&\quad +(g\hbar\omega_A\sigma_- a^\dagger - \hbar g\omega\sigma_- a^\dagger) + \hbar g^2[\sigma_+ a, \sigma_- a] \\
&= 0
\end{aligned} \tag{5}
$$

(3) There are two sets of eigen states $|e,n\rangle$ $|g,n\rangle$, the corresponding eigenvalues are

$$N|e,n\rangle = (a^\dagger a + \sigma_+\sigma_-)|e,n\rangle = (n+1)|e,n\rangle \tag{6}$$

$$N|g,n\rangle = (a^\dagger a + \sigma_+\sigma_-)|g,n\rangle = n|g,n\rangle \tag{7}$$

HW8b solution

(1) Recall Eq. (5.6.17) in the textbook, the zeroth order and first perturbations are:

$$c_n^{(0)}(t) = \delta_{ni} \tag{1}$$

$$\begin{aligned} c_n^{(1)}(t) &= \frac{-i}{\hbar}\int_{t_0}^{t} \langle n|V_i(t')|i\rangle dt' \\ &= \frac{-i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ni}t'} V_{ni}(t')dt' \end{aligned} \tag{2}$$

then, substituting the potential in this problem $V(t) = \lambda\cos(\omega t)\sigma_x$ into the zeroth order and first order perturbation, we obtain

$$c_g^{(0)}(t) = 0 \tag{3}$$

$$c_e^{(0)}(t) = 1 \tag{4}$$

$$\begin{aligned} c_e^{(1)} &= \frac{-i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ee}t'} V_{ee}(t')dt' \\ &= \frac{-i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ee}t'} 0 dt' \\ &= 0 \end{aligned} \tag{5}$$

$$\begin{aligned} c_g^{(1)} &= \frac{-i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ge}t'} V_{ge}(t')dt' \\ &= \frac{-i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ge}t'} \lambda\cos(\omega t')dt' \\ &= -\frac{i\lambda}{2\hbar}\int_{0}^{t} dt'[e^{i(\omega_A+\omega)t'} + e^{i(\omega_A-\omega)t'}] \\ &= -\frac{\lambda}{2\hbar}[\frac{e^{i(\omega_A+\omega)t}-1}{\omega_A+\omega} + \frac{e^{i(\omega_A-\omega)t}-1}{\omega_A-\omega}] \end{aligned} \tag{6}$$

These are the zeroth order and first order approximations.

(2) First, use the relation $E - V(x) = 0$ to determine the two turning points as

$$E = V(x) = \frac{1}{2}\omega^2 m x^2 \tag{7}$$

So,

$$x_1 = -\sqrt{\frac{2E}{m\omega^2}} \tag{8}$$

$$x_2 = \sqrt{\frac{2E}{m\omega^2}} \tag{9}$$

Then substituting the result to the integral, we find

$$\int_{x_1}^{x_2} dx \sqrt{2m(E - \frac{1}{2}\omega^2 m x^2)} = \frac{\pi E}{\omega} \tag{10}$$

Therefore

$$\frac{\pi E}{\omega} = \hbar\pi(n + \frac{1}{2}) \tag{11}$$

Finally,

$$E_n = \hbar\omega(n + \frac{1}{2}) \tag{12}$$

This is the same result as we derived from the exact solution.

(3) Replacing the potential by $V = k|x|$, and following the same procedure, we obtain the energy levels in the V-shaped as

$$E_n = [\frac{3\pi}{4}(n + \frac{1}{2})]^{2/3}[\frac{e^2\hbar^2 E^2}{2m}]^{1/3} \tag{13}$$

HW9 solution

1. Sakurai, problem 3.1

   In order to find the eigenvalues and eigenvectors of σ_y, we solve the Secular equation

$$\det(\sigma_y - \lambda I) = 0 \tag{1}$$

Then, it is easy to find the eigenvalues are:

$$\lambda_1 = 1 \tag{2}$$

with the corresponding eigenvector

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} \tag{3}$$

and

$$\lambda_2 = -1 \tag{4}$$

with the corresponding eigenvector

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \tag{5}$$

The probatility of finding the state $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is in the $\hbar/2$ state is

$$|\langle\psi_1|\psi\rangle|^2 = \frac{1}{2}|i\alpha + \beta|^2 \tag{6}$$

2. Checking the commutation relation is a quite easy task, we will just give one example

here.

$$
\begin{aligned}
[J_x, J_y] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= iJ_z \qquad (7)
\end{aligned}
$$

Similarly, we can check all the other commutation relations.

3. The map $|\alpha\rangle \to D(R)|\alpha\rangle = |\alpha\rangle_R$ is surjection, so we can write $D(R)$ as

$$D(R) = \sum_n |\alpha_n\rangle_R \langle\alpha_n| \qquad (8)$$

$$
\begin{aligned}
D(R)^\dagger D(R) &= \sum_{mn} |\alpha_n\rangle\langle\alpha_n|_{RR}|\alpha_m\rangle\langle\alpha_m| \\
&= \sum_{mn} |\alpha_n\rangle\delta_{mn}\langle\alpha_m| \\
&= 1 \qquad (9)
\end{aligned}
$$

So, $D(R)$ is unitary.

HW10 solution

1. The Heisenberg equation of motion is

$$\begin{aligned}\frac{d}{dt}K_1 &= \frac{i}{\hbar}[H, K_1] \\ &= \frac{i}{2\hbar}[\frac{K_1^2}{I_1} + \frac{K_2^2}{I_2} + \frac{K_3^2}{I_3}, K_1] \\ &= \frac{i}{2\hbar}[\frac{i\hbar}{I_2} - \frac{i\hbar}{I_3}]\{K_2, K_3\} \\ &= \frac{I_2 - I_3}{2I_2I_3}\{K_2, K_3\} \qquad (1)\end{aligned}$$

where we use

$$[K_1, K_2] = -i\hbar K_3 \qquad (2)$$

$$\begin{aligned}[K_2^2, K_1] &= K_2K_2K_1 - K_1K_2K_2 \\ &= K_2(K_1K_2 + i\hbar K_3) - (-K_2K_1 - i\hbar K_3)K_2 \\ &= i\hbar K_2K_3 + i\hbar K_3K_2 \qquad (3)\end{aligned}$$

$$\begin{aligned}[K_3^2, K_1] &= K_3K_3K_1 - K_1K_3K_3 \\ &= K_3(K_1K_3 - i\hbar K_2) - (K_3K_1 + i\hbar K_2)K_3 \\ &= -i\hbar K_2K_3 - i\hbar K_3K_2 \qquad (4)\end{aligned}$$

Similarly, we have

$$\frac{d}{dt}K_2 = \frac{i}{\hbar}[H, K_2] = \frac{I_3 - I_1}{2I_1I_3}\{K_1, K_3\} \qquad (5)$$

$$\frac{d}{dt}K_3 = \frac{i}{\hbar}[H, K_3] = \frac{I_1 - I_2}{2I_1I_2}\{K_1, K_2\} \qquad (6)$$

In classical case, K_1, K_2, K_3 commute with each other, then we have e.g.,

$$\begin{aligned}\frac{d}{dt}K_1 &= \frac{I_2 - I_3}{2I_2I_3}(K_2K_3 + K_3K_2) \\ &= \frac{I_2 - I_3}{2I_2I_3}2K_2K_3 \\ &= \frac{I_2 - I_3}{I_2I_3}I_2\omega_2I_3\omega_3 \\ &= (I_2 - I_3)\omega_2\omega_3 \qquad (7)\end{aligned}$$

Thus, we successfully reproduce Euler's equation.

2. Suppose $\hat{A}$ commutes with $\hat{J}_i$ and $\hat{J}_j$, then

$$\begin{aligned}
[\hat{A}, \hat{J}_k] &= [\hat{A}, -i\epsilon_{ijk}[\hat{J}_i, \hat{J}_j]] \\
&= -i\epsilon_{ijk}[\hat{A}, (\hat{J}_i\hat{J}_j - \hat{J}_j\hat{J}_i)] \\
&= -i\epsilon_{ijk}\{[\hat{A}, \hat{J}_i\hat{J}_j] - [\hat{A}, \hat{J}_j\hat{J}_i)]\} \\
&= 0 \qquad (8)
\end{aligned}$$

3. (1) Expand $e^{i\theta\sigma_z}$ as

$$\begin{aligned}
e^{i\theta\sigma_z} &= \sum_n \frac{1}{n!}(i\theta\sigma_z)^n \\
&= \sum_{n=odd} \frac{1}{n!}(i\theta\sigma_z)^n + \sum_{n=even} \frac{1}{n!}(i\theta\sigma_z)^n \\
&= \sum_{m=0}^{\infty} \frac{(i\theta)^{2m+1}}{(2m+1)!}(\sigma_z)^{2m+1} + \sum_{m=0}^{\infty} \frac{(i\theta)^{2m}}{(2m)!}(\sigma_z)^{2m} \\
&= \sum_{m=0}^{\infty} \frac{i(-1)^m\theta^{2m+1}}{(2m+1)!}\sigma_z + \sum_{m=0}^{\infty} \frac{(-1)^m\theta^{2m}}{(2m)!}I \\
&= i\sigma_z \sin\theta + I\cos\theta \qquad (9)
\end{aligned}$$

(2) According to (1),

$$\begin{aligned}
e^{\frac{i}{2}\theta\sigma_z} S_x e^{-\frac{i}{2}\theta\sigma_z} &= (i\sigma_z \sin\frac{\theta}{2} + \cos\frac{\theta}{2})S_x(-i\sigma_z \sin\frac{\theta}{2} + \cos\frac{\theta}{2}) \\
&= \sin^2\frac{\theta}{2}\sigma_z(\frac{\hbar}{2}\sigma_x)\sigma_z + i\frac{\hbar}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}[\sigma_z, \sigma_x] + \frac{\hbar}{2}\cos^2\frac{\theta}{2}\sigma_x \\
&= -\frac{\hbar}{2}\sin^2\frac{\theta}{2}\sigma_x + \frac{\hbar}{2}\cos^2\frac{\theta}{2}\sigma_x - \frac{\hbar}{2}\sigma_y \sin\theta \\
&= \frac{\hbar}{2}\sigma_x \cos\theta - \frac{\hbar}{2}\sigma_y \sin\theta \qquad (10)
\end{aligned}$$