A STOCHASTIC APPROACH TO ENTANGLEMENT DYNAMICS OF QUBIT SYSTEMS

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ABSTRACT

The study of open quantum systems becomes crucial to understanding entanglement dynamics of two-level systems: no qubits are truly isolated from their surroundings. Local noisy environments inevitably cause the fast decoherence and disentanglement of qubits, a key hindrance in realizing the promising applications in quantum computing and quantum communication. In this thesis, the evolution of bipartite entanglement in the presence of noise is studied for two very fundamental systems: (1) two spatially separated qubits interacting with their local environment, a basic model of quantum communication, and (2) two uncoupled qubits interacting with a common environment, most relevant for qubits in quantum computing. Both models are fundamental for understanding the interworking of entangled qubits as they lose amplitude and phase information about their quantum state to the environment.

First, by taking advantage of the nonlocality of entanglement, a statistical correlation is introduced between the local noisy environments of distant qubits, causing the qubits to perceive themselves in a common environment where entanglement is preserved for a wide range of initial states. It will be shown that bipartite entanglement can in fact be modulated by correlated local noise for both dephasing and dissipative fields and the conditions for providing a full statistical correlation in the non-Markovian regime will be explored. Secondly, for the first time the dynamical equations for two qubits in a common environment are solved for exactly in the dephasing and dissipative cases, shedding much light on what conditions are optimal for maintaining a high level of entanglement, such as symmetry of the qubits and memory of the environment. Lastly, the fast-tracking of entanglement will be discussed for the first time with respect to non-Markovian systems. It is proposed that calculating the average entanglement evolution over individual trajectories of the qubits, which are stochastic due to fluctuations from the noise, can provide a reasonably good indication for the general trends of the actual entanglement as an upperbound. For certain initial states, the mean entanglement trajectories predicted an almost identical evolution of the exact entanglement, making it extremely useful for future approximate calculations of very complex multipartite systems.

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To Sebastiano.

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Chapter 1

Introduction

More than a century ago, the emergence of quantum theory shook the foundation of physics, which continues to reverberate today. One of the defining factors that truly separates the quantum realm from what is witnessed in classical physics is the intrinsic correlation between quantum particles known as entanglement. Stretching from popular science to the forefront of new technology, entanglement theory has spawned an abundance of profound applications and esoteric interpretations, shattering the common view of what is scientifically possible. Although the formalism for studying entanglement has developed into many beautiful complex techniques, the root of this phenomenon lies at the very center of quantum theory.

1.1 Closed Quantum Systems

In the general framework of quantum mechanics, the probabilistic description of the state of a system is contained in the wavefunction, $|\Psi_t\rangle$, which dynamically evolves through time according to the standard time-dependent Schrödinger Equation,

$$i\hbar\frac{\partial |\Psi_t\rangle}{\partial t} = \hat{H}_{Sys} |\Psi_t\rangle . \tag{1.1}$$

The energy information about the system is contained in the Hamiltonian operator, $\hat{H}_{Sys}$, which acts on the state vector $|\Psi_t\rangle$ and extracts the dynamical properties of the system. The beauty of Schrödinger's Equation is that such a concise equation has the ability to describe an extremely large system consisting of thousands of particles or simply that of a single object. The main focus of this work is on two-level quantum particles, often termed a "qubit" for its potential to serve as a quantum bit in the highly anticipated quantum computer [1]. The most common two-level systems are the ground and excited electronic energy levels of an atom, $|g\rangle$ and $|e\rangle$, or the spin states of a spin-$\frac{1}{2}$ particle, $|+\rangle$ and $|-\rangle$, the latter being the general model system adopted in this report.

If the system of interest is essentially isolated from all other external factors, this type of closed quantum system typically follows a unitary evolution, facilitating a trivial solution to equation (1.1) of the form[1]

$$|\Psi_t\rangle = e^{\frac{-i}{\hbar}\int_0^t dt' \hat{H}_{Sys}(t')} |\Psi_0\rangle \,, \tag{1.2}$$

This type of system dynamics have been studied extensively [3, 4, 5], divulging such unique properties as superposition states and quantum coherence, which are unfamiliar to the classical world. For instance, the quantum-mechanical state of a single spin in a closed system exists in a superposition of both energy levels and is represented by the pure state

$$|\Psi\rangle = a\,|+\rangle + b\,|-\rangle \,, \tag{1.3}$$

where $|a|^2$ is the probability of being in the spin-up state $|+\rangle$ and $|b|^2$ is the probability of being in the spin-down state $|-\rangle$. Because the qubit is isolated from all other factors, the wavefunction will follow a fixed evolution according to the Schrödinger Equation (1.2), in which it will remain in the pure superposition state throughout time. An analogous representation of this closed system is via the density operator, $\rho = |\Psi\rangle\langle\Psi|$, expanded as:

$$\rho = \begin{pmatrix} |a|^2 & a^*b \\ ab^* & |b|^2 \end{pmatrix}. \tag{1.4}$$

[1] if $\hat{H}_{Sys}$ commutes with itself at all mixed time points [2].

The diagonal elements reflect the probability of being in one of the distinct spin states and the off-diagonal elements demonstrate the unique possibility for this particle to exist in both states at once. This quantum interference between states is known as coherence and is unparalleled in the classical world. Furthermore, extending this concept to multiple qubits brought quantum theory to a whole new level.

The fact that a wavefunction describing many quantum objects can also exhibit coherence between energy states is fascinating in its own right. For example, suppose there are two qubits A and B whose wavefunction spans the combined Hilbert space $\mathcal{H}^A \otimes \mathcal{H}^B$. The composite two-qubit state takes the general form:

$$|\Psi_{AB}\rangle = a\,|{+}{+}\rangle_{AB} + b\,|{+}{-}\rangle_{AB} + c\,|{-}{+}\rangle_{AB} + d\,|{-}{-}\rangle_{AB}\,, \tag{1.5}$$

reflecting all of the possible spin combinations that the two qubits can exist in. The ensemble of this system is then represented by the density matrix

$$\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}| = \begin{pmatrix} |a|^2 & a^*b & a^*c & a^*d \\ b^*a & |b|^2 & b^*c & b^*d \\ c^*a & c^*b & |c|^2 & c^*d \\ d^*a & d^*b & d^*c & |d|^2 \end{pmatrix}, \tag{1.6}$$

where the off-diagonal elements now reflect the quantum interference between the two-qubit states. Elements on the anti-diagonal quantify the ability for both qubits to exhibit single-qubit coherence, and the other off-diagonal elements reflect the possibility that one qubit is in a superposition state while the other is in a single spin state. The startling factor is that in all of the details contained in this description of quantum coherence, not once was it necessary to consider the position of the particles. This implies that uncoupled quantum objects can have a strong correlation even if they are on opposite ends of the universe. From this arose the notion of quantum entanglement, where for some quantum systems it is impossible to know full information about one particle without the knowledge of its entangled counterpart. This immediately sparked the field of quantum information science and made entanglement extremely valuable to harness.

Both coherence and entanglement strongly persist within a closed bipartite system, such that two noninteracting entangled qubits will remain entangled indefinitely. Moreover, if you expand the system to include multiple particles, they too will become highly entangled with the rest of the system, however at the expense of a lesser degree of entanglement between the two original qubits of interest. For that reason, entanglement becomes extremely vulnerable to factors external to the closed system, such as electromagnetic radiation consisting of very many photons, the general makeup of the device that the qubits are in having many vibrational modes, and especially the apparatus of measurement used to probe the qubits. In fact, it is quite rare that two qubits will be found completely isolated from any type of external disturbance and will almost always experience the fast decoherence and disentanglement due to the dissipation of energy [6]. It then becomes necessary to include the environment as part of the total system in order to appropriately describe the entanglement dynamics of the qubits [7, 8, 9, 10].

1.2 Open Quantum Systems

An open quantum system [11] spans the expanded Hilbert space, $\mathcal{H}^S \otimes \mathcal{H}^{Env}$, encompassing a system of interest, the qubits, which are either embedded in or part of a much larger system, refered to here as the environment. The interaction between the system and environment then allows for the transfer of energy and phase information, causing a fluctuation in the dynamics of the qubits as they attempt to equilibrate themselves with their surroundings. The energy of the open quantum system is represented by the total Hamiltonian operator

$$\hat{H}_{Tot} = \hat{H}_{Sys} + \hat{H}_{Env} + \hat{H}_{Int}, \tag{1.7}$$

which manifests the large influence the environment has on the dynamics of the qubits via their interaction, $\hat{H}_{Int}$. The total state of the open quantum system can be expanded in a combined basis of the system $\{\alpha_i^S\}$ and environment $\{\beta_j^{Env}\}$, such that

$$|\Psi_{Tot}(t)\rangle = \sum_{ij} c_{ij}(t) |\alpha_i^S\rangle \otimes |\beta_j^{Env}\rangle , \tag{1.8}$$

where it becomes apparent that the system is now completely intertwined with the environment. Although this total state $|\Psi_{Tot}\rangle$ will undergo a unitary evolution according to the time-dependent Schrödinger Equation (1.1)

$$i\hbar \frac{\partial |\Psi_{tot}\rangle}{\partial t} = (\hat{H}_{Sys} + \hat{H}_{Env} + \hat{H}_{Int}) |\Psi_{tot}\rangle , \tag{1.9}$$

the state of the qubits will no longer remain in a pure state as it did in the closed system. Due to the exchange of energy with the environment, the qubits have the possibility of evolving along various paths, resulting in an ensemble of states that is considered mixed. Such a mixed state implies that only partial information is known about the system, making it necessary to use the density matrix to properly describe the state of the system,

$$\rho_S = \sum_i p_i |\psi_i\rangle \langle\psi_i| , \tag{1.10}$$

having a distribution of probabilities p_i for being in possible pure state $|\psi_i\rangle$. In effect, the state of the qubits is a mathematical reduction of the Hilbert space of the total open quantum system to only include the system degrees of freedom, therefore averaging over all environmental factors, and is appropriately named the reduced density operator such that

$$\rho_S = Tr_{Env}\left[\left|\Psi_{tot}\right\rangle\left\langle\Psi_{tot}\right|\right]. \tag{1.11}$$

The dynamical equation governing the reduced density matrix is known as the master equation, which normally takes the standard form [4]

$$\dot{\rho_S} = \frac{-i}{\hbar}[\hat{H}_{Sys}, \rho_S] + \mathcal{L}(\rho). \tag{1.12}$$

The first term of the right-hand side clearly describes the free evolution of the qubits, making the essence of the master equation to define the super-operator $\mathcal{L}$ which encompasses all of the dissipation and fluctuating dynamics of the qubits in the open quantum system. The many techniques for deriving the reduced dynamics of the qubits will be presented in Chapter 2. Above all, from the evolution of the quantum state of the qubits comes the powerful knowledge of the entanglement dynamics of the model system.

1.3 Entanglement

1.31 Entanglement of Pure States

Although the idea of superposition states and quantum coherence were not the simplest notions to digest, there was enough evidence to validate these aspects of quantum theory, for instance in the photoelectric effect and Young's double slit experiment.

It was the nonlocality of the quantum interference between multipartite states that did not sit well with many. Once known as "spukhafte Fernwirkung" by Einstein, or the "spooky action at a distance," the idea of the nonlocal quantum correlation originated in a thought experiment put forth by Einstein, Podolsky, and Rosen as a way to show that quantum theory was incomplete [12]. The strange behavior described in the EPR paradox is such that a measurement taken on qubit A instantly reveals information about the quantum state of qubit B, even if system B was far enough away that the information traveling at the speed of light would reach that system much later than our ability to obtain knowledge of it. Before simply terming this type of state as unphysical, it was Schrödinger who argued that there is no true violation at hand and that the state is perfectly valid according to the superposition principle [5]. It was he who described such a state as 'entangled', directly referring to the fact that such a state could not be separated into two constituents represented by a tensor product, $|\psi\rangle_{AB} = |\psi_A\rangle \otimes |\psi_B\rangle$, known as a separable state. This state has the property that any measurement on system A would infer absolutely no information about system B, making them statistically independent states. Therefore, any composite two-qubit state that cannot be written as a separable tensor product must then be an entangled state.

In general, a vector in the composite space $\mathcal{H}_A \otimes \mathcal{H}_B$, having finite dimensions d_A and d_B respectively, can be expanded as

$$|\phi_{AB}\rangle = \sum_{i=1}^{r} g_i \, |i_A\rangle \otimes |i_B\rangle \tag{1.13}$$

where $\{|i_A\rangle\}$ and $\{|i_B\rangle\}$ are orthonormal sets in subspace A and B, respectively, and the coefficients g_i are non-negative scalars which uniquely determine vector $|\phi_{AB}\rangle$. This general theory in linear algebra is known as the Schmidt Decomposition of vector $|\phi_{AB}\rangle$ and provides much insight when applied to a bipartite quantum system [5]. When $r = 1$, all but one of the constants g_i is non-zero and the state vector is clearly separable into a tensor product such that

$$|\phi_{AB}\rangle = g_j |j_A\rangle \otimes |j_B\rangle . \tag{1.14}$$

Moreover, if $1 < r \leq min(d_A, d_B)$, the state can no longer be written as a tensor product state. We can therefore define an entangled state as any vector whose Schmidt rank, r, is greater than one.

From the definition of an entangled state as one which is not separable, it is apparent that there will be some states that exhibit a stronger quantum correlation between subsystems than others. How the degree of entanglement evolves in time becomes an important insight for the study of quantum systems, so defining a physically relevant measure of entanglement is crucial. All measures of entanglement must satisfy certain criteria [13], such that (1) the entanglement of independent systems is additive, (2) entanglement is conserved under a unitary transformation, (3) entanglement cannot be enhanced through any local operations on a subsystem, and (4) entanglement can be concentrated and diluted with unit asymptotic efficiency. For a pure state $|\psi\rangle$ describing a bipartite system that spans the combined Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the degree of entanglement is measured by the entropy of entanglement [13], $E(\psi) = S(\rho_A) = S(\rho_B)$, where $S(\rho) = -Tr\{\rho \log \rho\}$ is the von Neumann entropy and ρ_A and ρ_B are the reduced density matrices after taking a partial trace over the other subsystem's degrees of freedom, $\rho_A = Tr_B\{|\psi\rangle \langle\psi|\}$ and $\rho_B = Tr_A\{|\psi\rangle \langle\psi|\}$. This entropy of entanglement measures $E = 0$ for a separable state $\psi = \psi_A \otimes \psi_B$ and $E = 1$ for a maximally entangled state, such as the singlet state $\psi = \frac{1}{\sqrt{2}} (|+-\rangle - |-+\rangle)$.

From this definition, it is clear that the entropy of entanglement is a property of the ensemble state of the system. For this reason, when referring to the pure state evolution of two qubits in a closed quantum system, it is likely that all trajectories of the system follow along a uniform evolution, making its entanglement dynamics definable as a real-time measurement. In contrast, for a pair of qubits under the random effects of an external environment, their ensemble state represents a mixture of outcomes. Therefore, the interpretation of entanglement changes for qubits in an open quantum system and the calculation of entanglement of a mixed state must be treated in a different way.

1.32 Entanglement of a Mixed State

The nature of mixed state entanglement stems from the notion that a mixed state ρ, as in equation (1.10), can be unravelled into various pure-state decompositions, ψ_i, for which the measure of entanglement is calculable [13]. By then minimizing over all of the possible decompositions of the density matrix, a lowerbound is quantified for the amount of entanglement that is available to that ensemble system. For instance, Terhal and Horodecki [14] defined the Schmidt number, $r_S(\rho)$, as the lowest Schmidt rank, r, minimized over all possible unravellings of the density matrix,

$$r_S(\rho) = min \left\{ max_i \left[r(\psi_i) \right] \right\}. \tag{1.15}$$

Analogously, Bennett, DiVincenzo, Smolin, and Wootters defined the entanglement of formation as the average entanglement of pure states minimized over all decompositions of ρ [13],

$$E(\rho) = min \sum_i p_i E(\psi_i). \tag{1.16}$$

The entanglement of formation satisfies all of the necessary conditions to be considered a valid measurement of entanglement, however has the major drawback of being difficult to calculate. Wootters set out to define a measurement of entanglement that is equivalent to the entanglement of formation, but is less computationally exhaustive [15, 16].

For a pure or mixed bipartite state represented by ρ, $E(\rho) = \mathcal{E}(C(\rho))$, where $C(\rho) = Max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ is known as the Concurrence [16] and ranges from zero to one, zero pertaining to a separable state and one a maximally entangled state. Here, λ_i are non-negative real numbers for the square root of the eigenvalues of $\rho\tilde{\rho}$ in descending order where $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ represents the spin flip state. The function of Concurrence, $\mathcal{E}(C)$, is equal to the entanglement of formation and calculated as $\mathcal{E}(C) = h\left(\frac{1+\sqrt{1-C^2}}{2}\right)$ where h is the binary entropy function, $h(x) = -x\log x - (1-x)\log(1-x)$. Because the concurrence, $C(\rho)$, ranges from zero to one and is monotonically related to the entanglement of formation, it in itself is regarded as a measurement of entanglement and will be used throughout the remainder of the thesis because of its ease of calculation.

1.4 Applications based on Entanglement

At the forefront of new technology is the use of entangled qubits to perform faster operations that are known to us classically as well as completely new phenomena stemming from the nuances of quantum theory [17, 18, 19, 20, 21, 22, 23, 1, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]. In all proposed applications, a robust entanglement between the qubits is essential but is inevitably faced with the decoherence properties of the noisy environment in its surroundings. Here are a few major applications for which the study of open quantum systems becomes crucial.

1.41 Quantum teleportation

Much of quantum communication relies on sending information stored in a quantum state through a quantum communication channel that is subject to many noise factors [1]. In the case of quantum teleportation, quantum information is conveyed by simply sending classical messages between the communicators. Suppose two collaborators, Alice and Bob, at opposite ends of the world share a maximally entangled pair of qubits, each having one qubit in possession. The advantages of entanglement are already relevant such that when Alice measures her qubit she automatically knows the state of Bob's qubit without even having to measure it. Taking this phenomenon one step further, if Alice interacts her qubit with yet another arbitrary qubit, existing in its own superposition state $|\psi\rangle = \alpha\,|+\rangle + \beta\,|-\rangle$, Alice can measure both qubits and send that information classically to Bob. Based on that classical message received, Bob can accordingly make a measurement of his qubit and recover the original information of the superposition state of the arbitrary qubit. What is more, the quantum information was transferred to Bob across a great distance without wires and signals, as would be needed for a classical communication. It's as if the original quantum information stored in the superposition state of the third qubit has been teleported.

Without a robust control over entanglement, quantum teleportation becomes nearly impossible. The main problem occurring in this scheme is local noise acting on each qubit, which is known to irrevocably destroy entanglement. In Chapters 3 and 4, it is proposed that a statistical correlation between the noisy environments can in fact enhance the entanglement between distant qubits.

1.42 Quantum Cryptography

In all of the aforementioned scenarios, the fight against decoherence from external sources is eminent. This includes the fundamental idea that there exists no measurement which does not effect the quantum state it is measuring. In reality, this fact becomes extremely beneficial in terms of securing quantum communication channels [1]. The transfer of secure information generally relies on a shared key between communicators, which can be used to encode and decode messages. However, it is highly possible that an eavesdropper can obtain that key and gain access to the no-longer-secure information. Much doubt arises in how one can be sure that a communication channel is safe enough to send the key in the first place.

Suppose now that the two communicators, Alice and Bob, share a large number of EPR pairs, all having quantum state $|\psi\rangle = \frac{1}{\sqrt{2}}\{|00\rangle + |11\rangle\}$, where the notation of basis kets $|0\rangle$ and $|1\rangle$ generally alludes to binary code and the use of qubits in quantum computing. Alice can perform a measurement on her half of the shared qubits, revealing a string of zeroes and ones, which instantly collapses the quantum mechanical state of Bob's qubits into the same set of numbers. They can then compare their string to ensure that the quantum channel is clear. If, however, there is an eavesdropper trying to make a measurement of Alice's qubits and compromising the security of the channel, that measurement would subsequently change the quantum mechanical state of the qubit and cause Bob to produce a different string of numbers than what Alice originally found. Alice and Bob can then continue this procedure until they have verified the channel is secure and send their key. In quantum key distribution, the quantum measurement would uniquely act as a fail-proof way to identify unsafe communication, as long as the entanglement between the original EPR pairs can persist long enough.

1.43 The Quantum Computer: Qubits realized

The ability to simultaneously have information about multiple entangled qubits by simply measuring one qubit instantly sparked the idea of creating a computer based on quantum mechanics [1]. The substantial motivation for realizing the quantum computer is because of its promise for certain quantum algorithms to be exponentially faster than its classical analogue. By storing many pieces of classical information in a collection of entangled qubits, it can essentially be processed all at once by manipulating only a single qubit, all because of the coherent nature of the quantum state. This type of parallel computing, known as superdense coding [35], improves on many classical algorithms which tend to be very exhaustive, such as the quantum search algorithm [36] and the factoring of large prime numbers, a task that could greatly challenge cryptographic security schemes. While the motivation for building a working quantum computer is clear, there are many hindrances which stand in the way of this realization.

At the heart of the quantum computer is the assumption that the qubits remain in an entangled state long enough to perform the necessary quantum logic gates that make up a functioning quantum information processor. The three main components of a successful quantum operation are (1) preparation of the initial entangled state, (2) manipulation of a single qubit as well as multiple qubits without losing their coherent properties and (3) obtaining a readout of the final qubit state to be used as computer logic. There have been groundbreaking technologies that demonstrate great promise for the realization of the quantum computer, a few of which are named here.

Photon qubits are very useful in quantum computing due to the reasonable amount of control over them using optical fibers, beam splitters, and phase shifters.

The optical quantum computer makes use of the paired polarization states of the photon as a realization of a two-state quantum bit, such as horizontal and vertical polarization or clockwise and counter-clockwise circular polarization. In another manner, quantum computing based on cavity quantum electrodynamics (QED) utilizes the photon modes of an optical cavity as the superposition state used to store information [17, 33, 34]. One drawback for the mainstream use of this type of quantum computer is that scaling and making identical replications of this kind of qubit are quite difficult.

Another type of qubit is realized in the nuclear spin state of trapped ions [20, 19, 27], confined in space using electromagnetic fields and cooled so that the kinetic energy of the charged atom is much less than the hyperfine spin energy levels of the nucleus. Nuclear spins are strongly coupled through chemical bonds of the atoms and can be easily probed with monochromatic light to allow for active control of the qubit information. Although the techniques needed to manipulate nuclear spin qubits have been highly developed in the field of NMR spectroscopy, it is still very difficult to get an accurate readout signal because of noise.

Last, superconducting qubits provide a large benefit as a prototype of quantum computing because of its ready scalability [22, 21, 23, 24, 29, 25, 26, 28, 32, 31]. Given two superconducting islands linked through a Josephson junction, quantum information can be stored in a charge qubit [23, 22, 28, 29, 25], where one island either has no net charge or an excess of charge due to the tunneling of a Cooper pair. In another model, superconducting loops are used whose current is controllable through an external flux, making the direction of the current what defines this flux qubit state [21, 31]. The major hindrance that holds back this technology from mainstream application is the very dense environment that generally makes up superconductor devices, which greatly affects the coherent properties of the qubits. Ultimately, in all of the aforementioned open quantum systems, the common threat is the decoherence and disentanglement of the qubit state due to noisy environments.

1.5 Decoherence and Disentanglement in Open Quantum Systems

The vulnerability of coherence and entanglement in open quantum systems is simply the reaction of a very small system becoming overwhelmed by a much larger environment. The strong coherent properties known to two isolated qubits quickly decay as the qubits adjust to being a part of a larger whole. In mathematical terms, pure decoherence caused by fluctuations in the spacing of the qubit energy levels is marked by the decay of the off-diagonal elements in the reduced density matrix,

$$\rho_S(t) = \begin{pmatrix} |a|^2 & a^* b e^{-\Gamma t} \\ a b^* e^{-\Gamma t} & |b|^2 \end{pmatrix}, \tag{1.17}$$

where Γ represents the decoherence rate. Once diagonal, ρ_S contains only the classical sampling between two distinct states with no potential for a superposition. The timescale of full decoherence is $\tau_{dec} = \frac{1}{\Gamma}$ and depends on the specific model at hand and the type of decoherence mechanism.

A dephasing type of qubit-environment interaction results in the pure decoherence of the qubit state without any type of population transfer. In work by T. Yu and J. H. Eberly [37, 38], it was shown that the single-qubit dephasing time is always shorter than or equal to the two-qubit decoherence time, reflecting a cascade of effects within the decoherence mechanism. Phase noise typically causes the asymptotic decay of entanglement for qubits initially in an entangled pure state, however it was shown that disentanglement can occur faster than full two-qubit decoherence in some cases.

A dissipative type of interaction, resulting in the population transfer of the qubit states, is known to pose the strongest opposition to entanglement [39]. Most notably, amplitude noise causes qubits in a pure entangled state to disentangle in a finite time, coined as the sudden death of entanglement by Yu and Eberly [39, 10, 40, 41, 42]. This abrupt loss of control over entanglement has been witnessed experimentally by Almeida et al. [43], demonstrating the true hindering nature of dissipation.

It is quite valuable to theoretically analyze the entanglement dynamics of two-qubits within an open quantum system in order to understand the most beneficial scenarios for maintaining the coherent properties of the qubits. In the following chapters, many fundamental models will be analyzed under various environmental conditions with the aid of many well-developed theoretical techniques.

Chapter 2

Techniques

When treating a quantum mechanical problem in an open quantum system framework, the challenge arises in isolating the dynamics of the system of interest in order to highlight the entanglement of the qubits, as well as other properties of the system alone. The total Hamiltonian of the open quantum system expanded from Eq. (1.7) for any qubit system interacting with a bosonic bath is [4]

$$\begin{aligned} \hat{H}_{Tot} &= \hat{H}_{Sys} + \hat{H}_{Env} + \hat{H}_{Int} \\ &= \hat{H}_{Sys} + \sum_{\lambda} \omega_\lambda a_\lambda^\dagger a_\lambda + \sum_{\lambda} \left(g_\lambda^* L a_\lambda^\dagger + g_\lambda L^\dagger a_\lambda \right). \end{aligned} \tag{2.1}$$

This Hamiltonian operator holds true for any number of qubits described by $\hat{H}_{Sys}$ in the presence of a radiation field modeled as a collection of very many harmonic oscillators, each having a transition frequency ω_λ and the annihilation and creation operators that obey the commutation relation $[a_\lambda, a_\lambda^\dagger] = 1$. The essence of the model is contained in how the qubits interact with the environment via the Lindblad operator, L. Initially, the qubits are taken to interact with each and every mode of the radiation field with a specific coupling strength g_λ, however it will be shown that the collective effect of the environment on the system will be stochastic in nature.

Distinguishing the dynamics of the qubits from that of the total quantum system is at the core of theoretical problems dealing with open quantum systems. The evolution of the total wavefunction $|\Psi_{Tot}(t)\rangle$ is known from the classic Schrödinger equation approach in (1.9), facilitating a dynamical equation for the ensemble state of the open system, $\rho_{Tot}(t) = |\Psi_{Tot}(t)\rangle \langle \Psi_{Tot}(t)|$. It is then natural to trace out all environmental degrees of freedom in order to isolate the system dynamics alone, known as the reduced density matrix $\rho_S(t) = Tr_{Env}\{\rho_{Tot}(t)\}$, which is essential for the calculation of the qubit entanglement. Furthermore, resolving the time evolution for $\rho_S(t)$ allows for the calculation of qubit entanglement in a dynamic way, revealing much insight into the diffusive effects of a noisy environment on bipartite entanglement over time. Although there is no universal way of deriving the reduced density operator dynamics for all models, many useful methods have been developed. Some successful techniques with which many important models have been solved are the Feynman-Vernon Integral [11], projection operator techniques [2], Kraus operators [44], the master equation [45], and the quantum trajectory technique [46]. The latter two techniques are the the main focus here.

2.1 Master Equation

The most intuitive way to resolve the dynamics of the reduced density matrix is to simply derive a differential equation for it, called the master equation, which takes the general form

$$\dot{\rho_S} = \frac{-i}{\hbar}[\hat{H}_{Sys}, \rho_S] + \mathcal{L}(\rho), \tag{2.2}$$

where the commutator contains the free unitary evolution of the system and the superoperator $\mathcal{L}$ contains all information about amplitude and phase damping from environmental factors. The task of deriving the explicit form of the $\mathcal{L}$ operator is known to be extremely challenging for most complex systems, and as of yet does not exist for general non-Markovian environments. In actuality, there are only a few conditions which allow for an exact master equation to be derived.

For environments with a very short memory of past times, the Markov approximation becomes relevant [47]. It states that the value of the environmental noise at present time has no correlation to its past values, making the bath correlation function simply a sharply peaked delta-function. It was Lindblad who showed that if a linear master equation is Hermitian, preserves probability, has constant coefficients pertaining to its Markovian nature, and preserves positivity, it can always be written in the form [48, 49]

$$\dot{\rho}_S = -\frac{i}{\hbar}[\hat{H}_{Sys}, \rho_S] + \sum_j \left(L_j \rho_S L_j^\dagger - \frac{1}{2} L_j^\dagger L_j \rho_S - \frac{1}{2} \rho_S L_j^\dagger L_j \right). \tag{2.3}$$

This master equation is extremely powerful for describing the non-unitary evolution of qubits in a Markov environment and has become a standard for solving problems in many areas. For some special cases in the Markov regime, the solution to the master equation can be developed from Kraus operators [44], K_m,:

$$\rho_S(t) = \sum_m K_m^\dagger \rho_S(0) K_m, \tag{2.4}$$

where $\sum_m K_m K_m^\dagger = 1$. The simplest case is when the system dynamics are contained within a single Kraus operator, representing the unitary evolution of the system.

When the coupling between the system and environment is very weak, the Born approximation becomes relevant, allowing the system to be treated separately from the environment throughout time, $\rho_{Tot}(t) = \rho_S(t) \otimes \rho_{Env}$. Applying this approximation gives rise to the Redfield master equation, which is applicable for general bath correlation functions, even those in the non-Markovian regime. Beyond the Born approximation, there does not exist a viable master equation that generally accounts for exact memory effects for all models. The study of non-Markovian systems has become of great interest for photonic band-gap materials [30], quantum dots [50], atom lasers [51], and general non-Markovian quantum information processing [52, 53].

Therefore, techniques which bypass the need to derive the master equation become a very powerful tool, specifically the non-Markovian quantum trajectory technique.

2.2 Quantum Trajectory Technique

The second approach mentioned here takes advantage of the statistical nature of the qubits in the presence of the environment, which can be viewed collectively as simply a source of noise. It was shown that the reduced density operator can be unravelled into a collection of stochastically driven quantum state vectors, $|\psi_t(z)\rangle$, which live in the state space of the system and on average reconstruct the master equation dynamics, $\rho_S(t) = M[|\psi_t(z)\rangle \langle\psi_t(z)|]$, where $M[\cdot]$ denotes the statistical average over all possible realizations of the noise. Here, $|\psi_t(z)\rangle$ represents a single quantum trajectory of the pure state dynamics of the system under the influence of environmental noise, z_t. To put this into practice, the stochastic effects of noise on the evolution of this quantum state is contained in the Quantum State Diffusion (QSD) Equation developed by Gisin and Percival [7] and used countless times to resolve the qubit dynamics of many fundamental models [7, 54, 8, 55, 52, 56]. Under the conditions that the initial state of the open quantum system is in a separable state, $\rho_{Tot}(0) = \rho_S(0) \otimes \rho_{Env}(0)$, and the environment is at zero temperature, the explicit form of the QSD Equation for a general set of qubits is

$$\frac{\partial |\psi_t(z)\rangle}{\partial t} = -i\hat{H}_{Sys} |\psi_t(z)\rangle + L z_t |\psi_t(z)\rangle - L^\dagger \int_0^t ds M[z_t^* z_s] \frac{\delta |\psi_t(z)\rangle}{\delta z_s} \tag{2.5}$$

where L is the Lindblad operator describing the type of diffusive nature caused by the system-environment interaction and z_t is a complex Gaussian process satisfying $M[z_t] = 0$ and $M[z_t z_s] = 0$. The right hand side of Eq. (2.5) breaks down as the free unitary evolution of the system, the qubit interaction with the noisy environment, and the effects of environmental memory on the system. The memory of the environment is contained in the bath correlation function, $M[z_t^* z_s] \equiv \alpha(t,s)$, which quantifies how much the noise at time t depends on its values at past times s. It will be shown that various types of noise will have a significant effect on the dynamics of the qubits.

2.21 Noise and Environment Correlation Functions

Considering the quantization of electromagnetic radiation, the environment is modeled as a collection of a large number of harmonic oscillators, each interacting with the qubits in an ever-changing way. Instead of exactly accounting for each individual interaction between qubit and harmonic oscillator, the collective effect of the environment is taken into account, which is naturally quite random and represented by a Gaussian noise variable, z_t. In some cases, the environment is treated as as a classical field, for which the noise will be a real-valued variable. In other cases, the environment is fully quantized and the noise can be complex. Aside from satisfying the Gaussian distribution conditions, such that the mean of the noise is zero and the variance of the noise is one, the noise can also be characterized by its correlation function. In the case of white noise, the environment has no memory of past times and has an auto-correlation function that will only be nonzero at present time t, $M[z_t^* z_s] = \Gamma\delta(t-s)$, where Γ is the dissipation rate for amplitude noise or the dephasing rate for phase noise. This correlation function is representative of the Markov regime for which the correlation time of the environment is so short that memory effects can be ignored. In the case of non-Markovian noise, the current value of environmental noise depends on its past values, therefore adding memory effects to the dynamics of the total system. There are two common types of non-Markovian noise taken into account in this thesis.

Ornstein-Uhlenbeck Noise

The Ornstein-Uhlenbeck process was first introduced by Langevin to describe the velocity of a Brownian particle in a frictional substance [57, 58]. In general, it is quite useful for modeling a continuous random process which drifts toward a stationary long-term mean. Computationally, it has the benefit of the following closed-form description:

$$z(t+dt) = z(t) - \frac{1}{\tau}z(t)dt + c^{1/2}N(t)(dt)^{1/2} \tag{2.6}$$

where τ is the relaxation time, c is the diffusion constant, and $N(t)$ is a normal random variable. Ornstein-Uhlenbeck noise, although random, is affected by the relaxation mechanism, making future values depend on previous values, expressed in the autocorrelation function

$$M[z_t^* z_s] = \frac{\Gamma\gamma}{2}e^{-\gamma|t-s|}, \tag{2.7}$$

where s represents a time in the past. Here, Γ is the dissipation or dephasing rate of the system, depending on the type interaction mechanism, and γ determines the correlation time of the noise, $\tau_{corr} = \frac{1}{\gamma}$, and in turn controls the memory of the environment. For very small γ, non-Markovian effects become very prominent and will show to have a large influence on entanglement dynamics in Chapters 4 and 5. As γ becomes large, the effects of memory become weaker, approaching the autocorrelation function $M[z_t^* z_s] = \Gamma\delta(t-s)$, reflecting Markov noise. The use of Ornstein-Uhlenbeck noise is extremely advantageous for viewing the transition from non-Markov to Markov regimes. For many practical problems for which the accurate numerical simulation of the noise is difficult, the noise variable can be approximated as a collection of many Ornstein-Uhlenbeck noises, $\alpha(t-s) = \alpha_1(t-s)+\alpha_2(t-s)+...$, as an application of the central limit theorem [59].

Temperature-Dependent Noise

In the most realistic case, a major challenge to bipartite entanglement is noise from finite temperature reservoirs [60, 61, 62, 63, 37, 64, 65]. Bose *et al.* have demonstrated that a single qubit will become strongly entangled with a large thermal field [66, 60], implying a distraction from qubit-qubit entanglement in a finite temperature heat bath. A general temperature dependent bath correlation function in the continuous frequency domain is of the form [67]

$$\alpha(t-s) = 2\hbar \int_0^\infty d\omega J(\omega) \left[\coth\left(\frac{\hbar\omega}{2k_B T}\right) \cos(\omega(t-s)) + i \sin(\omega(t-s)) \right] \qquad (2.8)$$

where k_B is the Boltzmann constant and T is the temperature. The environment can be further modeled through the spectral density function, $J(\omega)$, for example as an Ohmic bath where $J(\omega) = \omega e^{-\omega/\omega_c}$, characterizing an environment that responds instantly. It will be shown in Chapter 4 that the temperature of the environment greatly impacts the entanglement dynamics of the qubits.

Correlated Noise

Besides the noise having a correlation to its own past values, it is also possible for two environments to have a dependence on each other's past time values. Suppose there are two environments present, having respective noise variables x_t and y_t. They each have an autocorrelation function described as $M[x_t^* x_s] = \alpha(t-s)$ and $M[y_t^* y_s] = \beta(t-s)$, but can also have a cross-correlation function $M[x_t^* y_s] = \gamma(t-s)$, such that the current value of noise x_t depends on the past values of noise y_t. It will be proposed in Chapters 3 and 4 that a statistical correlation between environments can simulate a common bath to very distant qubits and provide beneficial entanglement conditions for certain qubit schemes.

2.3 Solving the QSD Equation: The O-operator Ansatz

The large challenge of solving the QSD Equation in an exact way is evaluating the functional derivative with respect to noise, $\frac{\delta |\psi_t\rangle}{\delta z_s}$. One very effective way is to assume that this derivative can be replaced by a linear time-local operator, termed the O-operator, such that $\frac{\delta |\psi_t\rangle}{\delta z_s} = \hat{O}(t,s,z)\,|\psi\rangle$ with the initial condition $\hat{O}(t=s,s,z) = L$ [52, 56]. A closed set of differential equations for this operator can be uniquely derived from the consistency conditions:

$$\frac{\delta}{\delta z_s}\left(\frac{\partial |\psi_t\rangle}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\delta |\psi_t\rangle}{\delta z_s}\right). \tag{2.9}$$

in which case the O-operator would facilitate an exact solution to the QSD equation. However, for certain types of interactions with a non-Markovian environment, the derivation of explicit equations governing the O-operator is not possible, for which is becomes necessary to employ a few approximations.

In the event that the system and environment are weakly coupled, the O-operator can be expanded in powers of the coupling constant,

$$O(t,s,z) = \sum_n g^n O_n(t,s,z) \tag{2.10}$$

where $O_n(t,s,z)$ are powers of L in the interaction picture of the system. The lowest order of this weak coupling approximation is therefore

$$O(t,s,z) \equiv O_0(t,s,z) = e^{-i\hat{H}_{Sys}(t-s)} L e^{i\hat{H}_{Sys}(t-s)}, \tag{2.11}$$

which, when applied to the derivation of the master equation, recovers the Redfield master equation.

For general coupling strengths, it is often of good practice to expand the O-operator in powers of the delay time $(t-s)$ [68]:

$$\hat{O}(t,s,z) \approx \hat{O}(t=s) + \left(\frac{\partial \hat{O}(t,s,z)}{\partial t}\right)_{t=s} (t-s) + ... \tag{2.12}$$

The simplest approximation is the first term in the series, $O(t,s,z) \approx L$, another formulation of the Markov approximation because it is most accurate when used in the case of a memoryless environment. It is always possible to derive a Markov master equation, but it does not always properly describe more complicated systems. For near-Markov environments, the second order term can be taken into account in what is known as the Post-Markov approximation. Although it provides an improvement beyond the Markov approximation, it does not properly account for true memory effects, especially regarding entanglement, as will be shown in Chapter 5.

2.31 Non-linear QSD Equation

After evaluating the O-operator, $O(t,s,z)$, the QSD equation (2.5) becomes a linear partial differential equation which can be solved numerically with ease. However, linear stochastic differential equations generally have a large problem preserving the norm of the wavefunction. At the extremes, the norm can be either zero or infinity, making it necessary to have a very large sampling in order to properly recover the correct statistics, $M[\langle\psi_t(z)|\psi_t(z)\rangle] = 1$. This poses a computational problem for more complicated system, for which it would take an enormously large number of realizations of the quantum state to produce physically accurate results. Instead, it becomes convenient to define a state vector that is normalized at every point in time,

$$|\tilde{\psi}_t\rangle = \frac{|\psi_t\rangle}{||\,\langle\psi_t|\psi_t\rangle\,||}, \tag{2.13}$$

along with a subsequent QSD equation that always preserves the norm of the quantum state:

$$\begin{aligned} \frac{\partial\,|\tilde{\psi}_t\rangle}{\partial t} &= -i\hat{H}_{Sys}\,|\tilde{\psi}_t\rangle + (L - \langle L\rangle_t)\tilde{z}_t\,|\tilde{\psi}_t\rangle \\ &- \int_0^t ds\alpha(t-s)[(L^\dagger - \langle L^\dagger\rangle_t)\hat{O}(t,s,\tilde{z}) \\ &- \langle (L^\dagger - \langle L^\dagger\rangle_t)\hat{O}(t,s,\tilde{z})\rangle_t]\,|\tilde{\psi}_t\rangle \end{aligned} \tag{2.14}$$

with the shifted noise variable

$$\tilde{z}_t = z_t + \int_0^t ds\alpha^*(t-s)\,\langle L^\dagger\rangle_s \tag{2.15}$$

where $\langle A\rangle_t = \langle\tilde{\psi}_t|A|\tilde{\psi}_t\rangle$ is the ensemble mean of operator A and $\alpha(t-s)$ is the autocorrelation function of the noise. Although this equation is now nonlinear and more difficult to solve individually, it is of good practice to adopt this QSD equation to ensure the preservation of the norm on every trajectory.

2.4 Consistency between approaches

Further analyzing the QSD Eq. (2.5) reveals the stochastic behavior governing the qubit dynamics. The qubits are no longer evolving along a fixed evolution as they would in a closed system, but rather evolve along many different trajectories according to the statistical properties of the noise. By computationally generating a realization of the random noise $z_t^{(i)}$, the QSD equation can be solved for a specific stochastic evolution of the qubits, $|\psi_t^{(i)}\rangle$, known as a quantum trajectory. This solution would represent a real time measurement of the system dynamics, but would not provide a substantial amount of information about what to expect in future trajectories or give insight into how to control the entanglement of the qubits. By producing a large number of quantum trajectories, N, we can recover the evolution of the ensemble dynamics of the qubits governed by the reduced density matrix operator, $\rho_S(t) = M[|\psi_t(z)\rangle\langle\psi_t(z)|]$, by numerically averaging the probability density over all realizations at time point t,

$$\rho_S(t) = \frac{1}{N}\sum_{i}^{N} |\psi_t^{(i)}\rangle\langle\psi_t^{(i)}| \,. \tag{2.16}$$

Essentially, this accounts for the random environmental effects in an average way, which is consistent with the implications of the master equation.

In the following chapters, the entanglement dynamics for various fundamental qubit systems will be developed using both exact master equation and exact quantum trajectory techniques. In some cases, the exact entanglement evolution will be revealed for the very first time, divulging brand new insight into the strange behavior of entanglement.

Chapter 3

Semiclassical Model with Correlated White Noise

The realization of quantum communication schemes promises fail-proof security through quantum cryptography and unique features such as quantum teleportation [1]. The most basic setup, as depicted in Figure 3.1, consists of two distant entangled qubits A and B that each interact with their local environment, E_A and E_B respectively. Whether it be electromagnetic radiation, the modes of a cavity, or simply the composition of the device that the qubits are in, local environmental noise will undoubtedly cause the swift decay of entanglement between the qubits, leaving most scenarios ineffective. There has been extensive work dedicated to the quantum entanglement dynamics of qubits in the presence of local noise, both in the classical and quantum regimes [38, 39, 69, 70, 71, 72, 73, 74, 75, 76, 77]. Yu and Eberly demonstrated that for the local dephasing channel, qubits were found to disentangle in a shorter time than their individual dephasing times [38, 70]. They later showed that two initially entangled two-level atoms in spatially separated non-interacting cavities will also exhibit shorter nonlocal disentanglement times than the local decoherence times [39]. Both of these examples highlight the fragility of qubit entanglement when confronted with a local noisy environment, limiting the practicality of current quantum communication apparatuses.

It is strictly enforced that no local operation on a qubit can enhance the measure of entanglement, narrowing down the possibilities for robust entanglement control. The only way to modulate entanglement in our favor is then to augment a global factor in the open quantum system. It is known that a dephasing environment that is common to the qubits preserves the entanglement of a special class of initial states,

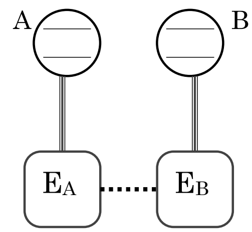

Figure 3.1: Two uncoupled qubits A and B separately coupled to classical stochastic fields, $E_A(t)$ and $E_B(t)$, which are then correlated.

called the decoherence-free subspace [78, 79]. Due to the nonlocality of entanglement, a qubit cannot necessarily sense the amount of distance between itself and its entangled counterpart, presenting a golden opportunity to mimic a common environment that preserves entanglement for distant qubits. This becomes a statement of high symmetry, ensuring that the local environmental conditions are nearly identical, as well as a proposition of correlated noise. By introducing a statistical correlation between the local noise variables, the separate environments will be perceived as one common environment by the qubits, allowing for the modulation of entanglement [73]. The full effect of correlated noise will be revealed in the study of entanglement dynamics between the two qubits for both the dephasing and dissipative models. In this chapter, it will be shown that entanglement can be fully preserved for a wide range of initial states via correlated classical noise.

3.1 Phase relaxation

When the decoherence of the quantum state of the qubit is solely phase relaxation, the population of the energy levels of the quantum particle are not affected by the interaction with the environment. This dephasing interaction represents a rotation of the Bloch vector about the $\hat{z}$-axis, therefore only changing the phase of the basis states $\{|-\rangle, |+\rangle\}$. A specific case is the Pauli-Z matrix, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which simply represents no change in the $|-\rangle$ state of the qubit and a π phase shift of the $|+\rangle$ state. Supposing that boths qubits interact with uncorrelated local phase noise, the decoherence of the superposition state would normally lead to an exponential decay or even the finite death of entanglement [38, 70, 77, 76, 75], which can now be completely preserved at its initial value under fully correlated noise, for certain initial states. The effective Hamiltonian for the two qubits A and B in the presence of classical phase noises, $f_A(t)$ and $f_B(t)$ respectively, is (setting $\hbar = 1$)

$$\hat{H}_{eff} = \hat{H}_{sys} + f_A(t)\sigma_z^A + f_B(t)\sigma_z^B \tag{3.1}$$

where the system Hamiltonian is explicitly $\hat{H}_{sys} = \frac{\omega_A}{2}\sigma_z^A + \frac{\omega_B}{2}\sigma_z^B$. The local noise variables obey the statistical properties

$$\begin{aligned} M[f_A(t)f_A(s)] &= \gamma_A\delta(t-s), \quad (3.2)\\ M[f_B(t)f_B(s)] &= \gamma_B\delta(t-s), \quad (3.3)\\ M[f_A(t)f_B(s)] &= \Gamma\delta(t-s), \quad (3.4) \end{aligned}$$

where $M[\cdot]$ denotes the ensemble average over all possible realizations of the classical noises. The parameters γ_A and γ_B are the rate of phase relaxation from the local noises, and for simplicity, they are taken to be equal such that $\gamma_A = \gamma_B = \gamma$. The parameter Γ then determines the cross-correlation of the two fields, $M[f_A(t)f_B(s)]$, and is restricted by the relation $\Gamma \leq \gamma$. Due to the random effects of the classical noises, the quantum state of the qubits, $|\psi\rangle$, will undergo a stochastic unitary evolution,

$$\frac{\partial |\psi\rangle}{\partial t} = -i\left(H_{sys} + f_A(t)\sigma_z^A + f_B(t)\sigma_z^B\right)|\psi\rangle\,. \tag{3.5}$$

The properties of the environment will play a pivotal role in determining the entanglement possibilities available to certain qubit states.

In order to resolve the ensemble dynamics of the qubits, one must average over all outcomes of the random behavior by taking the mean of the probability density over all possible sets of noise, $\rho_S = M[|\psi\rangle\langle\psi|]$, known as the reduced density operator. The time dependence of this operator is then expressed as $\dot{\rho_S} = M[|\dot{\psi}\rangle\langle\psi|] + M[|\psi\rangle\langle\dot{\psi}|]$, where the stochastic Schrödinger Equation (3.5) is inserted as

$$M[|\dot{\psi}\rangle\langle\psi|] = -iH_{sys}\rho_S - i\sigma_z^A M[f_A(t)|\psi\rangle\langle\psi|] - i\sigma_z^B M[f_B(t)|\psi\rangle\langle\psi|]. \tag{3.6}$$

The property of correlated noise is then applied within the mean noise terms:

$$\begin{aligned} M[f_A(t)|\psi\rangle\langle\psi|] &= \int_0^t ds M[f_A(t)f_A(s)]M\left[\frac{\delta|\psi\rangle\langle\psi|}{\delta f_A(s)}\right] \\ &+ \int_0^t ds M[f_A(t)f_B(s)]M\left[\frac{\delta|\psi\rangle\langle\psi|}{\delta f_B(s)}\right], \end{aligned} \tag{3.7}$$

$$\begin{aligned} M[f_B(t)|\psi\rangle\langle\psi|] &= \int_0^t ds M[f_B(t)f_A(s)]M\left[\frac{\delta|\psi\rangle\langle\psi|}{\delta f_A(s)}\right] \\ &+ \int_0^t ds M[f_B(t)f_B(s)]M\left[\frac{\delta|\psi\rangle\langle\psi|}{\delta f_B(s)}\right]. \end{aligned} \tag{3.8}$$

Applying the chain rule, the functional derivatives are expanded as

$$\frac{\delta|\psi\rangle\langle\psi|}{\delta f_A(s)} = \frac{\delta|\psi\rangle}{\delta f_A(s)}\langle\psi| + |\psi\rangle\frac{\delta\langle\psi|}{\delta f_A(s)}, \tag{3.9}$$

$$\frac{\delta|\psi\rangle\langle\psi|}{\delta f_B(s)} = \frac{\delta|\psi\rangle}{\delta f_B(s)}\langle\psi| + |\psi\rangle\frac{\delta\langle\psi|}{\delta f_B(s)}. \tag{3.10}$$

A unique ansatz is presented in Chapter 2, Section 2.3, for replacing the functional derivatives in the above equations by a linear operator, termed the O-operator, which, in general, is a function of time and noise. Because the environments have no memory of past times, the Markov approximation is employed and the O-operators take on a very simple form:

$$\frac{\delta \left|\psi\right\rangle}{\delta f_A(s)} = \hat{O}_A(t,s,f_A,f_B)\left|\psi\right\rangle = i\sigma_z^A\left|\psi\right\rangle \tag{3.11}$$

$$\frac{\delta \left\langle\psi\right|}{\delta f_A(s)} = \left\langle\psi\right|\hat{O}_A^\dagger(t,s,f_A,f_B) = -\left\langle\psi\right| i\sigma_z^A \tag{3.12}$$

$$\frac{\delta \left|\psi\right\rangle}{\delta f_B(s)} = \hat{O}_B(t,s,f_A,f_B)\left|\psi\right\rangle = i\sigma_z^B\left|\psi\right\rangle \tag{3.13}$$

$$\frac{\delta \left\langle\psi\right|}{\delta f_B(s)} = \left\langle\psi\right|\hat{O}_B^\dagger(t,s,f_A,f_B) = -\left\langle\psi\right| i\sigma_z^B \tag{3.14}$$

Subsequently, the master equation governing the ensemble dynamics of two qubits in the presence of correlated phase noise is

$$\begin{aligned}\dot{\rho_S} = -i[H_{sys},\rho_S] &- \gamma_A\left(\rho_S - \sigma_z^A\rho_S\sigma_z^A\right) - \gamma_B\left(\rho_S - \sigma_z^B\rho_S\sigma_z^B\right) \\ &- \Gamma\left(\sigma_z^A\sigma_z^B\rho_S + \rho_S\sigma_z^A\sigma_z^B - \sigma_z^A\rho_S\sigma_z^B - \sigma_z^B\rho_S\sigma_z^A\right).\end{aligned} \tag{3.15}$$

At a first glance, Eq. (3.15) immediately displays the relaxation of qubit A at rate γ_A, that of qubit B at rate γ_B, and the additional effects due to the cross-correlation between the two fields. Without correlated noise, $\Gamma = 0$, the master equation would simply be the sum of the two single-qubit Lindblad master equations [80], in which case the two qubits would evolve separately throughout time and their entanglement would inevitably deteriorate. The correlation between the two classical fields, characterized by various levels of Γ, will then either enhance or diminish the degree of entanglement of the entire evolution depending on the initial state of the qubits.

Since in most applications the main goal is to maintain the qubits in a maximally entangled state, much attention is placed on the four Bell States, $\left|\Psi^\pm\right\rangle = \frac{1}{\sqrt{2}}\{\left|{++}\right\rangle \pm \left|{--}\right\rangle\}$ and $\left|\Phi^\pm\right\rangle = \frac{1}{\sqrt{2}}\{\left|{+-}\right\rangle \pm \left|{-+}\right\rangle\}$, which all have a concurrence measurement of 1. In addition, the ability to improve the entanglement of a more general initial state will arise with a special X-form density matrix,

$$\rho_a = \frac{1}{2}\begin{pmatrix} a & 0 & 0 & a \\ 0 & (1-a) & (1-a) & 0 \\ 0 & (1-a) & (1-a) & 0 \\ a & 0 & 0 & a \end{pmatrix}, \tag{3.16}$$

where $0 \leq a \leq 1$. This state includes the Bell states as special cases, $a = 0$ referring to state $|\Psi^+\rangle$ and $a = 1$ to state $|\Phi^+\rangle$. The initial entanglement value of this X-form matrix as a function of parameter a, as measured by concurrence [16], is $C(\rho_a) = 2\max\{0, |2a-1|\}$, accounting for a large range of initial entanglement conditions. It will be revealed that for some entangled states, correlated noise will stand to fight against decoherence and consequently cause a slower disentanglement, whereas for others, correlated noise can add to the decoherence mechanism and cause a faster decay of entanglement.

Only in the dephasing case does one find an analytical solution to the two-qubit master equation,

$$\rho_S(t) = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0)e^{-2\gamma t - i\omega t} & \rho_{13}(0)e^{-2\gamma t - i\omega t} & \rho_{14}(0)e^{-4(\gamma+\Gamma)t - 2i\omega t} \\ \rho_{21}(0)e^{-2\gamma t + i\omega t} & \rho_{22}(0) & \rho_{23}(0)e^{-4(\gamma-\Gamma)t} & \rho_{24}(0)e^{-2\gamma t - i\omega t} \\ \rho_{31}(0)e^{-2\gamma t + i\omega t} & \rho_{32}(0)e^{-4(\gamma-\Gamma)t} & \rho_{33}(0) & \rho_{34}(0)e^{-2\gamma t - i\omega t} \\ \rho_{41}(0)e^{-4(\gamma+\Gamma)t + 2i\omega t} & \rho_{42}(0)e^{-2\gamma t + i\omega t} & \rho_{43}(0)e^{-2\gamma t + i\omega t} & \rho_{44}(0) \end{pmatrix}. \tag{3.17}$$

From this solution it is shown that out of all reduced density matrix elements, correlated noise only effects the anti-diagonal elements, $\rho_{14}(t)$ and $\rho_{23}(t)$ and their conjugates. Therefore, initial states of the X-form in Eq. (3.16) are chosen to be analyzed,

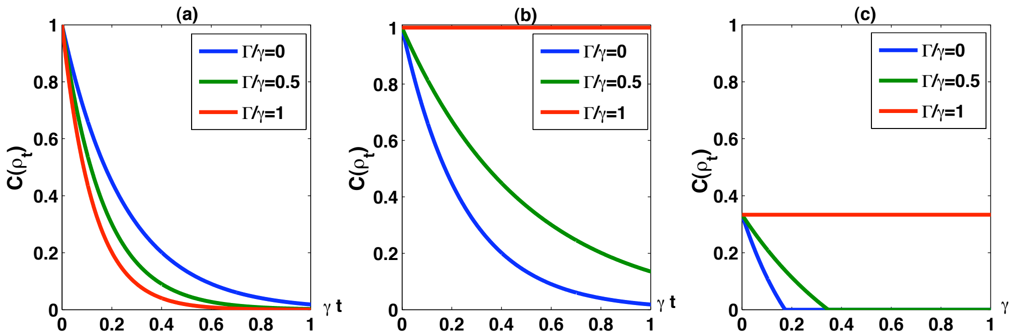

Figure 3.2: Entanglement evolution for qubits in the initial state (a)$|\Psi^{\pm}\rangle$, (b) $|\Phi^{\pm}\rangle$, and (c) the X-form matrix in Eq. (3.16) under a dephasing interaction with local baths having various levels of correlation governed by Γ/γ.

for which the concurrence throughout time will remain in the form

$$C(\rho_S(t)) = 2\max\{0, ae^{-4(\gamma+\Gamma)t} - (1-a), (1-a)e^{-4(\gamma-\Gamma)t} - a\}. \qquad (3.18)$$

The entanglement evolution is illustrated in Figure 3.2 for initial states (a) $|\Psi^{\pm}\rangle$, (b) $|\Phi^{\pm}\rangle$, and (c) the X-form matrix with $a = 1/3$, having an initial entanglement of $C(\rho_a(0)) = \frac{1}{3}$. The concurrence is plotted over the general timescale of the local dephasing time, γt, and the cross-correlation is then controlled by Γ/γ, which equals zero when the environments are statistically independent and one when they are fully correlated.

For all values of γ, when the environments are completely uncorrelated, $\Gamma/\gamma = 0$, the pure states $|\Psi^{\pm}\rangle$ and $|\Phi^{\pm}\rangle$ exhibit the exponential decay of entanglement, well-known to pure state phase relaxation [38, 10], and the mixed X-state experiences the sudden death of entanglement [70]. However, correlated noise affects these states in different ways. On one hand, for the Bell State $|\Psi^{\pm}\rangle$, the correlation between the stochastic fields causes the concurrence curve to move downward, denoting a decrease in the degree of entanglement and a faster disentanglement rate. On the other hand, for Bell state $|\Phi^{\pm}\rangle$ and the X-form state, correlated noise markedly enhances the entanglement between the qubits. Remarkably, at the maximum level of correlation, $\Gamma/\gamma = 1$, the qubits in $|\Phi^{\pm}\rangle$ are able to remain indefinitely in the maximally entangled state. The class of special X-form states with $a < \frac{1}{2}$ can therefore be highlighted as being preserved at their initial entanglement value for all time due to correlated noise. This is also reflected in the analytical solution for the concurrence in equation (3.18), for which $a < \frac{1}{2}$ becomes steady at $C(\rho_a(t)) = 1 - 2a$ only when $\Gamma = \gamma$. A full analysis of the entanglement evolution for the spectrum of special X-form states under fully correlated noise is plotted in Figure 3.3.

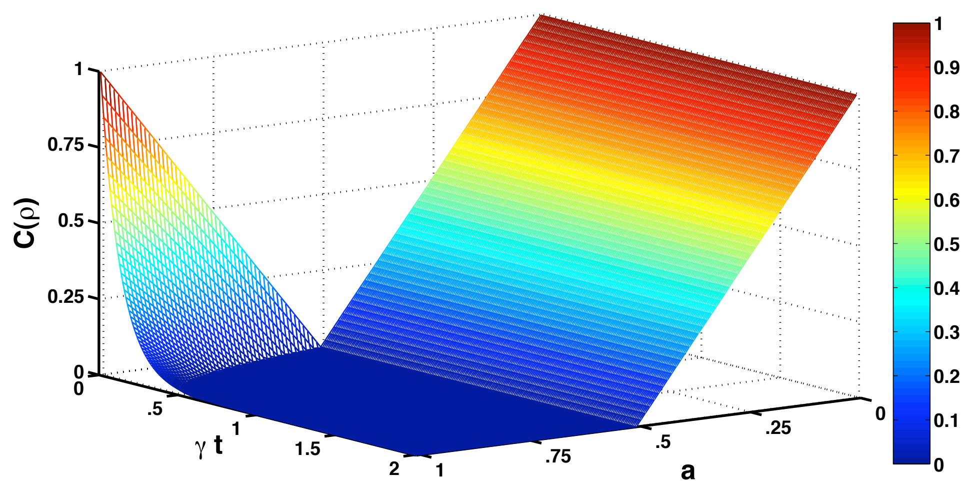

Figure 3.3: Entanglement evolution as a function of parameter a, denoting the spectrum of initial states in the special X-form matrix in Eq. (3.16) for the dephasing model with fully correlated local baths.

The right half of the plot portrays the complete preservation of entanglement via correlated noise for $a < \frac{1}{2}$, with the lower bound $a = 0$ illustrating the robust sustenance of entangled Bell state $|\Phi^{+}\rangle$. In constrast, when $a > \frac{1}{2}$, the sudden death of entanglement prevails, except for the case of $a = 0$, for which the entanglement of $|\Psi^{+}\rangle$

asymptotically decays to zero. In between these two cases lies the separable state, $a = \frac{1}{2}$, which is unaffected by the correlation of white phase noise. In Chapter 4, it will be shown that correlated noise at a finite temperature can generate entanglement between qubits, even for an initially separable state.

The exponential factor in the analytical solution for reduced density matrix element $\rho_{23}(t)$, $(\gamma - \Gamma)$, which governs the concurrence of the X-state when $a < \frac{1}{2}$, exactly demonstrates the idea of "subtracting" decoherence effects of the environment through correlated noise. Clearly, the ability to fully preserve entanglement between qubits via correlated white phase noise was demonstrated for a wide range of initial states. The results will be quite different for correlated amplitude noise.

3.2 Dissipation

A dissipative mechanism is a source of decoherence in which populations of the qubit states will change due to the exchange of energy with the environment. In terms of a qubit in the basis $\{|-\rangle, |+\rangle\}$, this will refer to population in the $|-\rangle$ state transferring to the $|+\rangle$ state and vice versa, which is normally represented by the raising and lowering operators, $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, which in combination form the Pauli spin X operator, $\sigma_x = \sigma_+ + \sigma_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This open quantum system represents an extension of the spin-boson model to the semi-classical regime, a well-known problem in condensed matter physics [81, 82, 63, 83]. The effective Hamiltonian representing two qubits in the presence of local amplitude noises, $g_A(t)$ and $g_B(t)$ is

$$\hat{H}_{eff} = \hat{H}_{sys} + g_A(t)\sigma_x^A + g_B(t)\sigma_x^B. \tag{3.19}$$

The classical noise variables have Markov statistical properties

$$
\begin{aligned}
M[g_A(t)g_A(s)] &= \chi_A\delta(t-s), && (3.20)\\
M[g_B(t)g_B(s)] &= \chi_B\delta(t-s), && (3.21)\\
M[g_A(t)g_B(s)] &= \Gamma\delta(t-s), && (3.22)
\end{aligned}
$$

where χ_A and χ_B are the individual dissipation rates of the qubits and are taken to be equal, $\chi_A = \chi_B = \chi$. Once again, Γ quantifies the level of correlation between the noise and must be less than or equal to χ in magnitude.

Similar steps can be taken to derive the Markov master equation as in Section 3.1. Following closely to Eqs. (3.5-3.15), the main difference will be in defining the functional derivatives with respect to amplitude noise:

$$
\begin{aligned}
\frac{\delta\left|\psi\right\rangle}{\delta g_A(s)} &= \hat{O}_A(t,s,g_A,g_B)\left|\psi\right\rangle = i\sigma_x^A\left|\psi\right\rangle && (3.23)\\
\frac{\delta\left\langle\psi\right|}{\delta g_A(s)} &= \left\langle\psi\right|\hat{O}_A^\dagger(t,s,g_A,g_B) = -\left\langle\psi\right| i\sigma_x^A && (3.24)\\
\frac{\delta\left|\psi\right\rangle}{\delta g_B(s)} &= \hat{O}_B(t,s,g_A,g_B)\left|\psi\right\rangle = i\sigma_x^B\left|\psi\right\rangle && (3.25)\\
\frac{\delta\left\langle\psi\right|}{\delta g_B(s)} &= \left\langle\psi\right|\hat{O}_B^\dagger(t,s,g_A,g_B) = -\left\langle\psi\right| i\sigma_x^B. && (3.26)
\end{aligned}
$$

Finally, the master equation for two qubits under correlated classical amplitude noise is

$$
\begin{aligned}
\dot{\rho_S} = -i[H_{sys},\rho_S] &- \chi_A\left(\rho_S - \sigma_x^A\rho_S\sigma_x^A\right) - \chi_B\left(\rho_S - \sigma_x^B\rho_S\sigma_x^B\right) && (3.27)\\
&- \Gamma\left(\sigma_x^A\sigma_x^B\rho_S + \rho_S\sigma_x^A\sigma_x^B - \sigma_x^A\rho_S\sigma_x^B - \sigma_x^B\rho_S\sigma_x^A\right).
\end{aligned}
$$

Although this master equation does not facilitate an analytical solution for the reduced density operator evolution, the entanglement dynamics were calculated numerically for the special X-form initial state of Eq. (3.16) and plotted against χt in Figure 3.4. For values of parameter a, sweeping through a wide range of initial entangle-

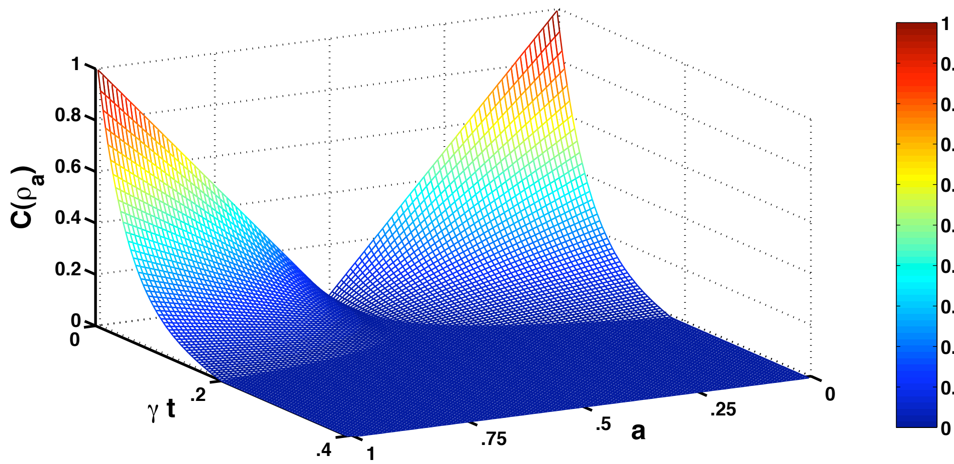

Figure 3.4: Entanglement evolution as a function of parameter a, denoting the spectrum of initial states in the special X-form matrix in Eq. (3.16) for the dissipative model with fully correlated local baths.

ment conditions, the effects of correlated amplitude noise all prove to be detrimental to entanglement. Whereas for a dephasing interaction the correlated noise was able to fight against the decoherence of certain qubit states, it has been shown only to assist in the fast decay of entanglement. The fate of all qubit states under correlated amplitude noise is the sudden death of entanglement.

3.3 Summary

Qubits in the presence of local classical noise will always experience decoherence and disentanglement as a consequence. By introducing full statistical correlation between the two environments, the entanglement can be modulated and sometimes even maintained at its maximum value, depending on the type of decoherence mechanism. For a dephasing type of system-environment interaction, causing only phase relaxation of the qubit state, a correlation between the noises counteracted the decoherence effect and provided a complete preservation of entanglement for all time for the Bell State $|\Phi^{\pm}\rangle$ and the X-form states with $a > \frac{1}{2}$. In contrast, correlated amplitude noise caused the sudden death of all entangled states, making dissipation the largest challenge against entanglement control. The true test of the open quantum system will be combating the effects of dissipative noise. A more general environment consists of both amplitude and phase noises and, in future work, the effects of correlated noise on this type of mixed decoherence system will be analyzed. Comparing the various relaxation timescales and their effects on entanglement with reveal more insight into how to generally control entanglement via correlated noise. In the next chapter, correlated noise will be derived from first principles in a fully quantized approach.

Chapter 4

Correlated Non-Markovian Phase Noise for Quantized Environments

In the previous chapter, the promising effects of correlated noise were revealed for two spatially separated qubits in the presence of local classical Markovian environments [73]. When the local noises were statistically dependent on one another, it provided an optimal scenario for entanglement preservation for a large range of entangled states. This leads to the task of defining correlated quantized noise on a fundamental level and exploring the realm of complex non-Markovian environments, such as Ornstein-Uhlenbeck noise and a thermal environment.

It is known that qubits in a common environment are able to remain fully entangled, even under phase relaxation, for a certain class of initial states living in the decoherence-free subspace (DFS) [78, 79]. However, this is technically only true in the highly symmetrical case that the qubits obey permutation symmetry. In reality, qubits in a common bath experience the environment differently, perhaps if the atoms are at different positions in the cavity. This will cause the qubits to have different couplings with the modes of the cavity and hence different noise variables[1] that are statistically correlated. In the perfectly symmetrical case, the noises are fully correlated and it is this complete statistical dependence between the noises that provides the means for entanglement to be fully preserved [37, 38]. Therefore, if we introduce a statistical correlation between the local noises of two qubits in separate cavities, we can in effect modulate their measure of entanglement, as was shown in previous work for classical environments [73]. This leads to the task of defining correlated quantized noise on a fundamental level and exploring the realm

[1] A quantum or classical environment is referred to as noise when its collective effect on the qubit is stochastic in nature and can be represented by a random variable.

of complex non-Markovian environments, such as Ornstein-Uhlenbeck noise and a thermal environment. In these cases, it will be necessary for the cavities to provide identical environmental conditions for both qubits in order for them to have the permutation symmetry necessary for the complete preservation of entanglement. This requires the qubits to have equal coupling strengths to each mode as well as the cavities both having the same mean number of photons in each mode. Although this is a very restrictive criteria, it will be shown that even a small statistical correlation between the cavity noises will provide an improvement to entanglement for a wide range of initial qubit states.

Because entanglement is a nonlocal phenomena, it doesn't matter whether the qubits are nanometers or miles apart. Suppose there are two spatially separated atoms, each in a cavity providing local phase noise, as depicted in Figure 4.1. If the cavities provide identical environmental conditions, such as a uniform structure and identical modes, and the local noise variables are then fully correlated, perhaps if the cavities are connected by an optical fiber, the qubits will perceive themselves to be in a common environment where entanglement does not decay. In this way, correlated phase noise can provide a great improvement to quantum communication schemes, where the loss of quantum coherence is known to be a large problem.

4.1 Deriving the QSD Equation

In this chapter, the stochastic dynamics are developed for two atoms in spatially separate cavities, whose modes are modeled as a quantized environment. The most

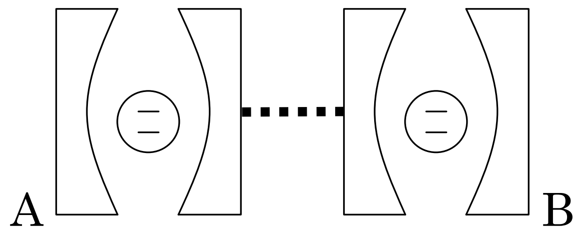

Figure 4.1: Two spatially separated qubits each in a cavity providing local phase noise. The cavities have identical modes and correlated noise, allowing for the entanglement between the qubits to be preserved.

general Hamiltonian describing this open quantum system is

$$H_{Tot} = H_{Sys} + H_{Env} + H_{Int} \tag{4.1}$$

$$H_{Sys} = \frac{\omega_A}{2}\sigma_z^A + \frac{\omega_B}{2}\sigma_z^B \tag{4.2}$$

$$H_{Env} = \sum_\lambda \omega_\lambda^{(A)} a_\lambda^\dagger a_\lambda + \sum_\lambda \omega_\lambda^{(B)} b_\lambda^\dagger b_\lambda \tag{4.3}$$

$$H_{Int} = \sum_\lambda \left(g_\lambda^* L_A a_\lambda^\dagger + g_\lambda L_A^\dagger a_\lambda \right) + \sum_\lambda \left(f_\lambda^* L_B b_\lambda^\dagger + f_\lambda L_B^\dagger b_\lambda \right). \tag{4.4}$$

The system Hamiltonian, $\hat{H}_{Sys}$, describes the energy of the two qubits, having transition frequencies ω_A and ω_B, respectively. The modes of the cavities, having respective frequencies $\omega_\lambda^{(A)}$ and $\omega_\lambda^{(B)}$, are modeled as a collection of harmonic oscillators in the environment Hamiltonian, $\hat{H}_{Env}$, and have respective creation operators a_λ and b_λ. The qubits then interact with their local environments in the interaction Hamiltonian, $\hat{H}_{Int}$, via the respective coupling constants, g_λ and f_λ, and the Lindblad operators, L_A and L_B, which describe the type of decoherence mechanism is caused by their cavity. Here, the focus is on phase relaxation where $L_A = \sigma_z^A$ and $L_B = \sigma_z^B$, but the Quantum State Diffusion (QSD) equation [7, 8, 49, 52] for the dynamics of the qubits will be derived for a general interaction.

In order to convince the qubits that they are in one single environment, the cavities must have the same number of modes with equivalent frequencies, $\omega_\lambda^{(A)} = \omega_\lambda^{(B)} = \omega_\lambda$. In the interaction picture of the environment, the time-dependent Schrödinger equation for the quantum state of the entire open quantum system, $|\Psi_{Tot}\rangle$, is

$$\begin{aligned} \frac{d\,|\Psi_{Tot}\rangle}{dt} = -i\hat{H}_{Sys}\,|\Psi_{Tot}\rangle &- iL_A \sum_\lambda g_\lambda a_\lambda e^{i\omega_\lambda t}\,|\Psi_{Tot}\rangle - iL_A^\dagger \sum_\lambda g_\lambda^* a_\lambda^\dagger e^{-i\omega_\lambda t}\,|\Psi_{Tot}\rangle \\ &- iL_B \sum_\lambda f_\lambda b_\lambda e^{i\omega_\lambda t}\,|\Psi_{Tot}\rangle - iL_B^\dagger \sum_\lambda f_\lambda^* b_\lambda^\dagger e^{-i\omega_\lambda t}\,|\Psi_{Tot}\rangle \end{aligned} \tag{4.5}$$

In order to resolve the dynamics of the qubits alone via the QSD equation, there must be a few assumptions made. First, the state of the open quantum system can be expanded into the tensor product of the qubit state, $|\psi_S\rangle$ and the state of the environments, $|z^{(A)}\rangle \otimes |z^{(B)}\rangle$, which are all considered to initially be separable. The environment states $|z^{(A)}\rangle$ and $|z^{(B)}\rangle$ are a representation of Bargmann coherent states for each mode, such that

$$|z^{(A)}\rangle = |z_1^{(A)}\rangle \otimes |z_2^{(A)}\rangle \otimes \cdots \otimes |z_\lambda^{(A)}\rangle \otimes \cdots \tag{4.6}$$

$$|z^{(B)}\rangle = |z_1^{(B)}\rangle \otimes |z_2^{(B)}\rangle \otimes \cdots \otimes |z_\lambda^{(B)}\rangle \otimes \cdots. \tag{4.7}$$

The second assumption is that all of the cavity modes begin in the vacuum state $|0_\lambda\rangle$. It follows that the total wavefunction in Eq. (4.5) can be expanded to $|\Psi_{Tot}\rangle = |\psi_S\rangle \otimes |z^{(A)}\rangle \otimes |z^{(B)}\rangle$, where the annihilation and creation operators act in their respective environment spaces such that

$$
\begin{aligned}
a_\lambda \left|z_\lambda^{(A)}\right\rangle &= z_\lambda^{(A)} \left|z_\lambda^{(A)}\right\rangle & (4.8)\\
a_\lambda^\dagger \left|z_\lambda^{(A)}\right\rangle &= \frac{\partial}{\partial z_\lambda^{(A)}} \left|z_\lambda^{(A)}\right\rangle & (4.9)\\
b_\lambda \left|z_\lambda^{(B)}\right\rangle &= z_\lambda^{(B)} \left|z_\lambda^{(B)}\right\rangle & (4.10)\\
b_\lambda^\dagger \left|z_\lambda^{(B)}\right\rangle &= \frac{\partial}{\partial z_\lambda^{(B)}} \left|z_\lambda^{(B)}\right\rangle . & (4.11)
\end{aligned}
$$

It should be recalled that there is only the potential for entanglement to be preserved when the qubits obey permutation symmetry, such that if they switched positions there would be no difference in their surroundings. The condition of identical environmental conditions is then demonstrated in setting $z_\lambda^{(A)} = z_\lambda^{(B)} = z_\lambda$, such that the mean number of photons per mode, $|z_\lambda^{(A)}|^2$ for cavity A and $|z_\lambda^{(B)}|^2$ for cavity B, are exactly the same. Taking this action, the differential equation for the dynamics of the qubits interacting with the modes of their local cavities is

$$
\begin{aligned}
\frac{\partial \left|\psi_S\right\rangle}{\partial t} = -i\hat{H}_{Sys} \left|\psi_S\right\rangle &- iL_A \sum_\lambda g_\lambda z_\lambda e^{i\omega_\lambda t} \left|\psi_S\right\rangle - iL_A^\dagger \sum_\lambda g_\lambda^* e^{-i\omega_\lambda t} \frac{\partial \left|\psi_S\right\rangle}{\partial z_\lambda} \\
&- iL_B \sum_\lambda f_\lambda z_\lambda e^{i\omega_\lambda t} \left|\psi_S\right\rangle - iL_B^\dagger \sum_\lambda f_\lambda^* e^{-i\omega_\lambda t} \frac{\partial \left|\psi_S\right\rangle}{\partial z_\lambda}
\end{aligned}
\tag{4.12}
$$

The above equation describes the exact dynamics of the qubits interacting with each and every mode of their local environment. For a large number of modes, however, it becomes very useful to look at the collective effect of those environments, which is random in nature, and is manifested in the Gaussian noise variables defined as

$$x_t \equiv -i \sum_\lambda g_\lambda z_\lambda e^{i\omega_\lambda t} \tag{4.13}$$

$$y_t \equiv -i \sum_\lambda f_\lambda z_\lambda e^{i\omega_\lambda t}. \tag{4.14}$$

The noises have mean zero, $\langle x_t \rangle = \langle y_t \rangle = 0$, and variance equal to one, $\langle x_t^2 \rangle - \langle x_t \rangle^2 = \langle y_t^2 \rangle - \langle y_t \rangle^2 = 1$, as well as having auto- and cross-correlation functions:

$$M[x_t^* x_s] = \int \frac{d^2 z}{\pi} e^{-|z|^2} x_t^* x_s = \sum_\lambda |g_\lambda|^2 e^{-i\omega_\lambda (t-s)} \tag{4.15}$$

$$M[y_t^* y_s] = \int \frac{d^2 z}{\pi} e^{-|z|^2} y_t^* y_s = \sum_\lambda |f_\lambda|^2 e^{-i\omega_\lambda (t-s)} \tag{4.16}$$

$$M[x_t^* y_s] = \int \frac{d^2 z}{\pi} e^{-|z|^2} x_t^* y_s = \sum_\lambda g_\lambda^* f_\lambda e^{-i\omega_\lambda (t-s)} \tag{4.17}$$

$$M[y_t^* x_s] = \int \frac{d^2 z}{\pi} e^{-|z|^2} y_t^* x_s = \sum_\lambda f_\lambda^* g_\lambda e^{-i\omega_\lambda (t-s)} \tag{4.18}$$

where $d^2 z = d^2 z_1 d^2 z_2 \cdots d^2 z_\lambda \cdots$ and $|z|^2 = |z_1|^2 + |z_2|^2 + \ldots + |z_\lambda|^2 \ldots$. It must be noted that it was the identical environmental conditions that allowed for the cross-correlation of the noises in Eqs. (4.17,4.18) due to the identity in the Bragmann coherent state representation,

$$\int \frac{d^2 z_\lambda}{\pi} e^{-|z_\lambda|^2} z_\lambda z_\lambda^* = 1. \tag{4.19}$$

In addition, the difference in coupling constants g_λ and f_λ will determine the level of correlation between the noises, such that when $g_\lambda = f_\lambda$ there is a complete statistical correlation. The effect of correlated noise will become apparent in the analysis of the entanglement dynamics of the distant qubits.

Introducing the noises will transform Eq. (4.12) into one that is stochastic, where the wavefunction will depend explicitly on the noises, $|\psi_S\rangle = |\psi_t(x,y)\rangle$. Exploiting this dependence, the partial derivatives with respect to the cavity modes z_λ can also be transformed according to the chain rule,

$$\frac{\partial |\psi_t(x,y)\rangle}{\partial z_\lambda} = \int_0^t ds \left(\frac{\delta |\psi_t(x,y)\rangle}{\delta x_s}\right)\left(\frac{\partial x_s}{\partial z_\lambda}\right) + \left(\frac{\delta |\psi_t(x,y)\rangle}{\delta y_s}\right)\left(\frac{\partial y_s}{\partial z_\lambda}\right). \tag{4.20}$$

From Eqs. (4.13) and (4.14), it is deduced that

$$\frac{\partial x_s}{\partial z_\lambda} = g_\lambda e^{i\omega_\lambda s} \quad , \quad \frac{\partial y_s}{\partial z_\lambda} = f_\lambda e^{i\omega_\lambda s} \tag{4.21}$$

facilitating the compact form of the QSD equation for qubits interacting with local cavities whose noise variables are statistically correlated,

$$\begin{aligned} \frac{\partial |\psi_t\rangle}{\partial t} = -i\hat{H}_{Sys} |\psi_t\rangle \quad &+ \quad L_A x_t |\psi_t\rangle - L_A^\dagger \int_0^t ds M[x_t^* x_s] \frac{\delta |\psi_t\rangle}{\delta x_s} + M[x_t^* y_s] \frac{\delta |\psi_t\rangle}{\delta y_s} \\ &+ \quad L_B y_t |\psi_t\rangle - L_B^\dagger \int_0^t ds M[y_t^* x_s] \frac{\delta |\psi_t\rangle}{\delta x_s} + M[y_t^* y_s] \frac{\delta |\psi_t\rangle}{\delta y_s} \end{aligned} \tag{4.22}$$

where the dependence on x_t and y_t is now implied in the notation $|\psi_t\rangle$. Equation (4.22) generally describes the time dynamics of two spatially separated qubits interacting with local noises that are statistically correlated. For the specific case of a dephasing type of qubit-environment interaction, where only phase information about the wavefunction of the qubits is lost to the environment, the Lindblad operators $L_A = \sigma_z^A$ and $L_B = \sigma_z^B$ are substituted in.

The intricacy of finding an exact solution to the stochastic differential Eq. (4.22) is apparent in its nonlocal dependence on time through the noise variables under the memory integral [54]. This complexity is addressed by replacing the functional derivative of the state vector with respect to noise by the $\mathcal{O}$-operators [52, 56, 68, 65], such that

$$\frac{\delta |\psi_t\rangle}{\delta x_s} = \hat{\mathcal{O}}_A(t,s,x,y)\,|\psi_t\rangle\,, \tag{4.23}$$
$$\frac{\delta |\psi_t\rangle}{\delta y_s} = \hat{\mathcal{O}}_B(t,s,x,y)\,|\psi_t\rangle\,. \tag{4.24}$$

This approach generally applies to all QSD models, however in most cases there is no explicit solution for the $\mathcal{O}$-operators, making it necessary to use certain approximations. For our current model, we derive the exact $\mathcal{O}$-operators, facilitating both an exact and solvable form of the QSD Eq. (4.22). The distinct conditions that determine the explicit form of the differential equations which govern the $\mathcal{O}$-operators are the initial conditions, $\hat{\mathcal{O}}_A(t=s,s,x,y)=\sigma_z^A$ and $\hat{\mathcal{O}}_B(t=s,s,x,y)=\sigma_z^B$, and the consistancy conditions

$$\frac{\partial}{\partial t}\left(\frac{\delta |\psi_t\rangle}{\delta x_s}\right) = \frac{\delta}{\delta x_s}\left(\frac{\partial |\psi_t\rangle}{\partial t}\right), \tag{4.25}$$
$$\frac{\partial}{\partial t}\left(\frac{\delta |\psi_t\rangle}{\delta y_s}\right) = \frac{\delta}{\delta y_s}\left(\frac{\partial |\psi_t\rangle}{\partial t}\right). \tag{4.26}$$

For this particular model, the dynamic equations for $\hat{\mathcal{O}}_A(t,s,x,y)$ and $\hat{\mathcal{O}}_B(t,s,x,y)$ can be found in Appendix B. Due to the commutativity of the Lindblad operators with the system Hamiltonian, $[\hat{H}_{Sys},L_A]=[\hat{H}_{Sys},L_B]=0$, the $\mathcal{O}$-operators take on the very simple form,

$$\hat{\mathcal{O}}_A(t,s,x,y)=\sigma_z^A \tag{4.27}$$
$$\hat{\mathcal{O}}_B(t,s,x,y)=\sigma_z^B. \tag{4.28}$$

Therefore, the QSD equation for qubits exhibiting phase relaxation under local correlated noise is

$$\begin{aligned}\frac{\partial\left|\psi_t\right\rangle}{\partial t} = -i\hat{H}_{Sys}\left|\psi_t\right\rangle &+ \sigma_z^A x_t\left|\psi_t\right\rangle - (\sigma_z^A)^2\int_0^t ds M[x_t^* x_s]\left|\psi_t\right\rangle \\ &+ \sigma_z^B y_t\left|\psi_t\right\rangle - (\sigma_z^B)^2\int_0^t ds M[y_t^* y_s]\left|\psi_t\right\rangle \\ &- \sigma_z^A\otimes\sigma_z^B\int_0^t ds\left\{M[x_t^* y_s] + M[y_t^* x_s]\right\}\left|\psi_t\right\rangle .\end{aligned} \tag{4.29}$$

The first two lines of Eq. (4.29) highlight the separate relaxation dynamics of the qubits while interacting with their local cavities. The third line epitomizes the effects of correlated noise. Although the qubits are initially uncoupled and possibly at a great distance apart, the statistical correlation between the local noise variables builds a bridge between the qubits, causing them to quantum mechanically interact via the operator $\sigma_z^A\otimes\sigma_z^B$. It is this induced coupling between the qubits that will consequently cause the enhancement of their entanglement, depending on the level of correlation.

4.11 Correlated Noise

In order to quantify the correlation, it is relevant to take an introspective look at the noise variables reiterated below

$$x_t \equiv -i\sum_\lambda g_\lambda z_\lambda e^{i\omega_\lambda t} \tag{4.13}$$

$$y_t \equiv -i\sum_\lambda f_\lambda z_\lambda e^{i\omega_\lambda t}. \tag{4.14}$$

Within the current formalism, the two factors contributing to the level of statistical dependence between the noises are the coupling constants, g_λ and f_λ, and the assertion that both cavities have the same mean number of photons, $|z_\lambda|^2$, in each mode. It can be said that if the qubits are coupled to every cavity mode in exactly the same way, $g_\lambda = f_\lambda$, then the noise variables x_t and y_t are fully correlated. This also implies that their autocorrelation functions are equal, $M[x_t^* x_s] = M[y_t^* y_s]$. If the noises are to stray away from being correlated, it means that the qubits will couple in a different way to each mode of the cavity. This can be introduced in the following way:

$$y_t = \kappa x_t + (1-\kappa) r_t \tag{4.30}$$

where r_t numerically represents a Gaussian noise variable that is entirely statistically independent from x_t, such that $M[x_t^* r_s] = 0$. The parameter κ now quantifies the amount of correlation between the noises, such that when $\kappa = 0$, the noise environment of qubit B is completely indepedent from that of qubit A, while when $\kappa = 1$, they are exactly identical. The autocorrelation function of y_t now becomes

$$M[y_t^* y_s] = \kappa^2 M[x_t^* x_s] + (1-\kappa)^2 M[r_t^* r_s] \tag{4.31}$$

which once again characterizes the effect of parameter κ.

Inserting this ansatz into the QSD equation (4.29) reveals

$$\begin{aligned}\frac{\partial \left|\psi_t\right\rangle}{\partial t} = -i\hat{H}_{Sys}\left|\psi_t\right\rangle \quad &+ \quad (\sigma_z^A + \kappa\sigma_z^B) x_t \left|\psi_t\right\rangle - (\sigma_z^A + \kappa\sigma_z^B)^2 \int_0^t ds M[x_t^* x_s] \left|\psi_t\right\rangle \\ &+ \quad (1-\kappa)\sigma_z^B r_t \left|\psi_t\right\rangle - (1-\kappa)^2 (\sigma_z^B)^2 \int_0^t ds M[r_t^* r_s] \left|\psi_t\right\rangle .\end{aligned} \tag{4.32}$$

This equation exactly achieves the goal of quantifying correlated noise with a single parameter. When $\kappa = 0$, Eq. (4.32) boils down to reflect the dynamics of two qubits interacting with statistically independent noises [65, 63]:

$$\begin{aligned}\frac{\partial |\psi_t\rangle}{\partial t} = -i\hat{H}_{Sys} |\psi_t\rangle &+ \sigma_z^A x_t |\psi\rangle - \int_0^t dsM[x_t^* x_s] |\psi_t\rangle \\ &+ \sigma_z^B r_t |\psi\rangle - \int_0^t dsM[r_t^* r_s] |\psi_t\rangle ,\end{aligned} \tag{4.33}$$

while when $\kappa = 1$, the noises become fully correlated and Eq. (4.32) transforms into two qubits that perceive themselves to be in a common bath [84]:

$$\frac{\partial |\psi_t\rangle}{\partial t} = -i\hat{H}_{Sys} |\psi_t\rangle + (\sigma_z^A + \sigma_z^B) x_t |\psi\rangle - (\sigma_z^A + \sigma_z^B)^2 \int_0^t dsM[x_t^* x_s] |\psi_t\rangle . \tag{4.34}$$

In the next section, the ensemble dynamics of this system will be developed from the differential equation (4.32), revealing the entanglement evolution of the qubits as a function of correlation strength, κ.

4.2 Deriving the Master Equation

As of yet, the stochastic dynamics of the qubits in the presence of quantized correlated noise have been developed by deriving the QSD equation. Up until now, the statistical properties of the noise have been kept general, but will later be evaluated for non-Markovian noise, such as Ornstein-Uhlenbeck noise or that of a thermal environment. The memory of the environment will be contained in the terms defined as $X(t) \equiv \int_0^t dsM[x_t^* x_s]$ and $R(t) \equiv \int_0^t dsM[r_t^* r_s]$ and will have a significant effect on the entanglement dynamics of the system. The ensemble dynamics of the qubits will be solved for via the reduced density operator, which averages over all possible realizations of noise, $\rho_S = M[|\psi_t\rangle \langle\psi_t|]$. Deriving the time dependence of this operator, the master equation, will provide the means for the calculation of the entanglement evolution.

Simply taking the derivative of the reduced density operator with respect to time, $\dot{\rho}_S = M[|\dot{\psi}_t\rangle\langle\psi_t|] + M[|\psi\rangle\langle\dot{\psi}|]$ and inserting the dynamic information from Eq. (4.32) results in the following master equation:

$$\begin{aligned}\dot{\rho}_S = -i[\hat{H}_{Sys}, \rho_S] &+ (\sigma_z^A + \kappa\sigma_z^B)M[x_t |\psi_t\rangle\langle\psi_t|] - (\sigma_z^A + \kappa\sigma_z^B)^2 X(t)\rho \\ &+ (1-\kappa)\sigma_z^B M[r_t |\psi_t\rangle\langle\psi_t|] - (1-\kappa)^2(\sigma_z^B)^2 R(t)\rho + H.C.\end{aligned} \tag{4.35}$$

Because x_t and r_t are statistically independent by definition, the mean noise terms are:

$$M[x_t |\psi_t\rangle\langle\psi_t|] = \int_0^t ds M[x_t x_s^*] M\left[\frac{\delta |\psi_t\rangle\langle\psi_t|}{\delta x_s^*}\right] \tag{4.36}$$

$$M[r_t |\psi_t\rangle\langle\psi_t|] = \int_0^t ds M[r_t r_s^*] M\left[\frac{\delta |\psi_t\rangle\langle\psi_t|}{\delta r_s^*}\right]. \tag{4.37}$$

In this new representation in terms of x_t and r_t, the functional derivatives will once again benefit from applying the $\mathcal{O}$-operator ansatz. By inspection of Eq. (4.32), it is apparent that the $\mathcal{O}$-operators will take on the form

$$\frac{\delta |\psi_t\rangle}{\delta x_s} = \hat{\mathcal{O}}_C(t,s,x,r)|\psi_t\rangle = (\sigma_z^A + \kappa\sigma_z^B)|\psi_t\rangle \tag{4.38}$$

$$\frac{\delta |\psi_t\rangle}{\delta r_s} = \hat{\mathcal{O}}_D(t,s,x,r)|\psi_t\rangle = (1-\kappa)\sigma_z^B|\psi_t\rangle. \tag{4.39}$$

as will be elaborated in Appendix B. Ultimately, the master equation governing the qubit dynamics in the presence of non-Markovian correlated noise is

$$\begin{aligned}\dot{\rho_S} = -i[\hat{H}_{Sys}, \rho_S] &+ 2Re\{X(t)\}(\sigma_z^A \rho_S \sigma_z^A - \rho_S) \\ &+ 2(\kappa^2 Re\{X(t)\} + (1-\kappa)^2 Re\{R(t)\})(\sigma_z^B \rho_S \sigma_z^B - \rho_S) \\ &+ 2\kappa Re\{X(t)\}(\sigma_z^A \rho_S \sigma_z^B + \sigma_z^B \rho_S \sigma_z^A) \\ &- 2\kappa X(t)\sigma_z^A \sigma_z^B \rho_S - 2\kappa X^*(t)\rho_S \sigma_z^A \sigma_z^B. \end{aligned} \tag{4.40}$$

It recovers the master equation for two qubits interacting with local noises that are statistically independent when $\kappa = 0$,

$$\begin{aligned}\dot{\rho_S} = -i[\hat{H}_{Sys}, \rho_S] &+ 2Re\{X(t)\}(\sigma_z^A \rho_S \sigma_z^A - \rho_S) \\ &+ 2Re\{R(t)\}(\sigma_z^B \rho_S \sigma_z^B - \rho_S) \end{aligned} \tag{4.41}$$

as well as the master equation for two qubits that feel as if they are in a common bath due to the full correlation of their local noises when $\kappa = 1$,

$$\begin{aligned}\dot{\rho_S} = -i[\hat{H}_{Sys}, \rho_S] &+ 2Re\{X(t)\}(\sigma_z^A \rho_S \sigma_z^A - \rho_S) + 2Re\{X(t)\}(\sigma_z^B \rho_S \sigma_z^B - \rho_S) \\ &+ 2Re\{X(t)\}(\sigma_z^A \rho_S \sigma_z^B + \sigma_z^B \rho_S \sigma_z^A) \\ &- 2X(t)\sigma_z^A \sigma_z^B \rho_S - 2X^*(t)\rho_S \sigma_z^A \sigma_z^B. \end{aligned} \tag{4.42}$$

In the following sections, the master equation in Eq. (4.40) will be solved for specific types of non-Markovian noise and the exact entanglement dynamics will be revealed.

In the following sections, the master equation in Eq. (4.40) will be solved for specific types of non-Markovian noise and the exact entanglement evolution will be revealed for the initial reduced density operator in the special X-form matrix, as recalled from Chapter 3,

$$\rho_a = \frac{1}{2}\begin{pmatrix} a & 0 & 0 & a \\ 0 & (1-a) & (1-a) & 0 \\ 0 & (1-a) & (1-a) & 0 \\ a & 0 & 0 & a \end{pmatrix}, \tag{3.16}$$

which represents a large spectrum of entangled states for $0 \leq a \leq 1$. The measure of entanglement used throughout is the Concurrence [16], $C(\rho)$, which ranges from zero to one and is quite useful for its ease of calculation. The initial state in Eq. (3.16) represents a range of entangled states, $C(\rho_a) = Max\{0, |2a-1|\}$, which will be shown in the following section to be greatly affected by correlated noise.

4.21 Results: Ornstein-Uhlenbeck Noise

Characterizing environmental noise as an Ornstein-Uhlenbeck process pronounces it to be a continuous random variable which drifts toward a long-term mean value, satisfying an exponentially decaying correlation function, $\alpha(t-s) = \frac{\gamma}{2}e^{-\gamma|t-s|}$ [57, 58]. The parameter γ controls the correlation time of the environment and consequently impacts the memory of the system dynamics as well. In the limit that γ goes to infinity, results in the Markov regime are recovered, which coincide with the semi-classical correlated white noise model presented in Chapter 3. Here, the focus is on non-Markovian noise for which the correlation time of the environment becomes quite long and memory effects become strong, impacting the system entanglement, as depicted in Figure 4.2 for the maximally entanglement Bell state $|\Phi^{\pm}\rangle$ under phase relaxation from $\kappa = 0.5$ correlated Ornstein-Uhlenbeck noises. When γ is large, the concurrence curve reflects the rapid decay of entanglement that is reminiscent of

Markovian environments. The non-Markovian case, $\gamma = 0.1$ causes the entanglement to decay more slowly and nearly doubles the time it takes the qubits to disentangle, making it evident why studies of non-Markovian systems are quite relevant.

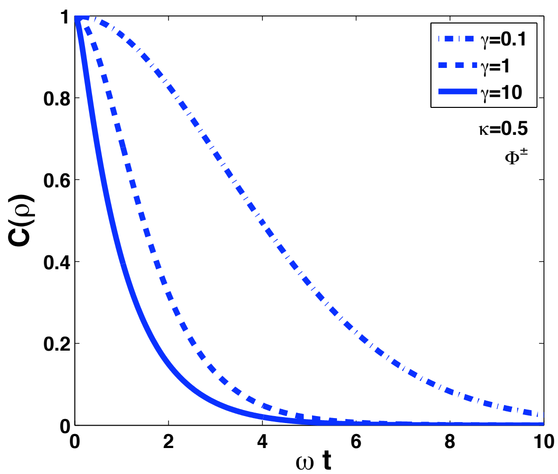

Figure 4.2: Entanglement dynamics of the Bell state $|\Phi^{\pm}\rangle$ for various autocorrelation times governed by the memory parameter γ. The different effects of Markovian and non-Markovian entanglement evolutions are portrayed.

In Figure 4.3, the entanglement dynamics for various initial states are plotted on the timescale of the system dynamics, ωt, for various levels of correlated noise governed by parameter κ. It depicts a very similar result to that of the semiclassical model with Markovian environments presented in Chapter 3. Although correlated noise does not improve entanglement conditions for the $|\Psi^{\pm}\rangle$ state, it remarkably preserves the entanglement of the the Bell state $|\Phi^{+}\rangle$ as well as the X-form matrix with $a = 1/3$ when the environments were fully correlated, $\kappa = 1$. As the environments drift farther away from full statistical correlation, the effects of local decoherence gradually begin to take over, resulting in the eventual death of entanglement. For entangled states with $a < \frac{1}{2}$, noise that is almost fully correlated allows the qubits to remain at a high level of entanglement at times much longer than the system's

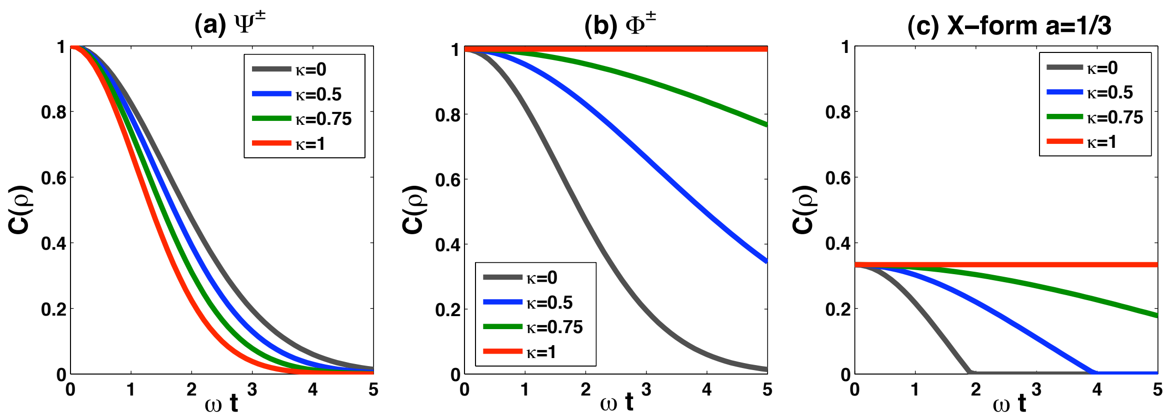

Figure 4.3: Entanglement dynamics for qubits interacting with correlated Ornstein-Uhlenbeck noise. The various effects of correlated noise are displayed by varying the correlation parameter, κ.

dynamic timescale, as illustrates in Figure 4.4. This pronounces correlated noise as a potential solution against the disentanglement of qubits in a non-Markovian dephasing environment. In the next section, unique effects will arise with a complex bath correlation function that is also temperature-dependent.

4.22 Results: Temperature-Dependent Ohmic Bath

One of the most common causes of decoherence in an open quantum system is frictional effects from a thermal reservoir. Bose *et al.* demonstrated a growth of entanglement between a single qubit and a large thermal field [66], implying a further distraction from qubit-qubit entanglement. The inevitable fate of entangled qubits in the presence of local thermal baths is the sudden death of entanglement [39, 40, 41, 62, 72]. However, two originally separable qubits in a common thermal field were shown to gain a small amount of entanglement [60, 85, 86], showing great promise for the positive influence of correlated noise. A general temperature dependent bath correlation

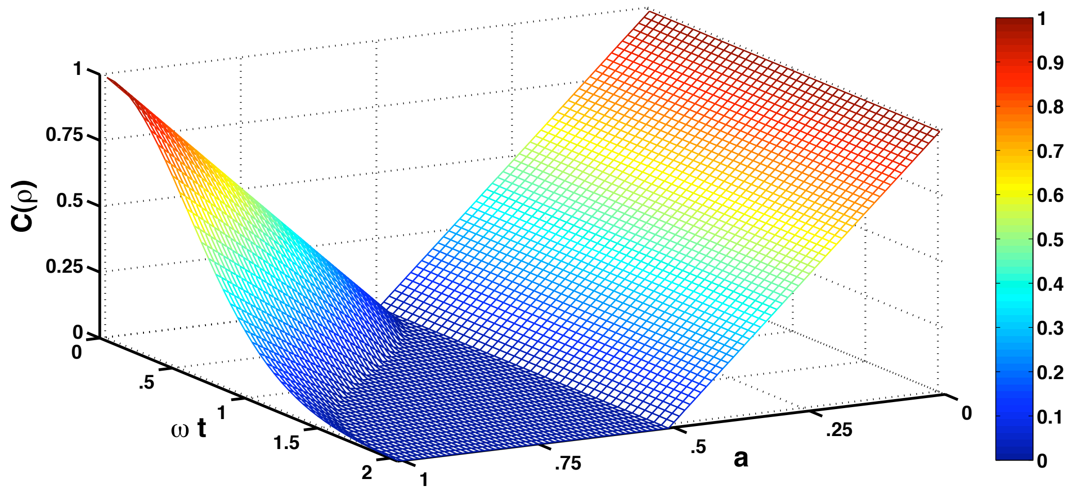

Figure 4.4: For fully correlated Ornstein-Uhlenbeck noise, $\kappa = 1$, the entanglement dynamics are revealed for the spectrum of initial states in the X-form matrix (3.16) as a function of parameter a. The class of entangled states described by $a < \frac{1}{2}$ is protected by correlated noise.

function in the continuous frequency domain is of the form [55, 46, 81]

$$\alpha(t,s) = 2\hbar \int_0^\infty d\omega J(\omega) \left[\coth\left(\frac{\hbar\omega}{2k_B T}\right)\cos(\omega(t-s)) + i\sin(\omega(t-s))\right] \tag{4.43}$$

where $k_B T$ is the thermal energy of the environment and $J(\omega)$ is the spectral density. The environment can be further modeled as an Ohmic bath [82], having a spectral density of the form $J(\omega) = \omega e^{-\omega/\omega_c}$. This represents a linear dependence on the frequency of the field, which is only an accurate model at very low frequencies that are much smaller than some cutoff frequency, ω_c, so the decaying function $e^{-\omega/\omega_c}$ imposes this low frequency condition. The parameter β, commonly defined as $\beta = \frac{1}{k_B T}$, will reflect the temperature dependence of the entanglement in Figures 4.5 and 4.6.

Generally, low temperatures, $\beta = 10$, are more pleasing to entanglement compared to high temperatures, $\beta = 0.1$, as depicted in Figure 4.5. A high temperature

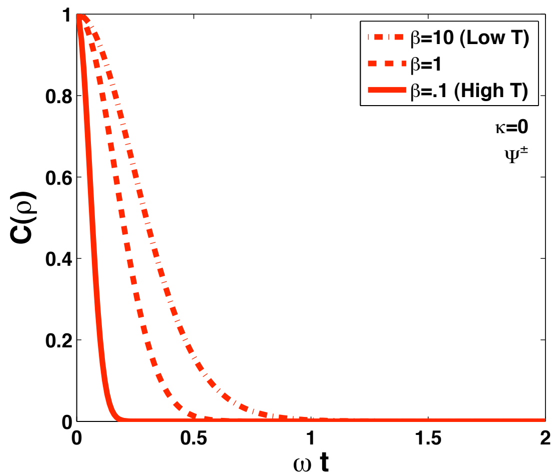

Figure 4.5: Entanglement dynamics of the Bell state $|\Psi^{\pm}\rangle$ for various temperatures governed by the parameter β. Low temperatures are revealed to be most beneficial for the measure of entanglement.

bath is associated with more rapid varying noise, which would more aggressively cause the decoherence of the composite qubit state and a disentanglement in a time much shorter than the timescale of the system dynamics. Based on this notion, the entanglement evolution for various levels of correlated temperature-dependent noise are calculated and plotted in Figure 4.6 for β fixed at $\beta = 10$.

The effects of temperature dependent noise are quite similar to that of Ornstein-Uhlenbeck noise, making correlated noise quite generally a good recipe for enhancing entanglement between qubits. It should be noted, however, that local thermal environments cause the qubits to disentangle in a much shorter time than Ornstein-Uhlenbeck noise. In fact, even at the very low thermal energy $\beta = 0.1$, the qubits still tend to disentangle on the timescale of ω or even shorter. The necessity for correlated noise then becomes evident for the state $|\Phi^{+}\rangle$, which is able to remain maximally entangled for all time. Special features arise from the complex noise that

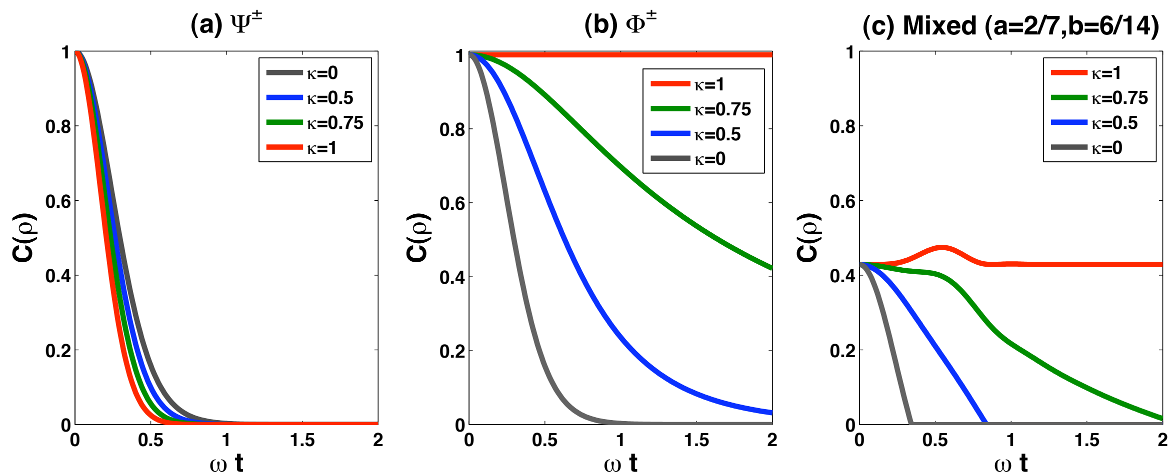

Figure 4.6: Entanglement dynamics for qubits interacting with correlated Ohmic thermal baths at low temperature, $\beta = 10$. The various effects of correlated noise are displayed by varying the correlation parameter, κ, and even present the generation of entanglement beyond its initial value for certain mixed states.

become apparent for off-diagonal terms b in the general initial state of the form

$$\rho_{mixed}(0) = \frac{1}{2}\begin{pmatrix} a & b & b & a \\ b & (1-a) & (1-a) & b \\ b & (1-a) & (1-a) & b \\ a & b & b & a \end{pmatrix}. \tag{4.44}$$

In order to maintain positivity of the reduced density matrix, the parameter b must obey the constraints $b \leq \sqrt{a(1-a)}$. The entanglement dynamics for the mixed state (4.44) with specific values $a = \frac{2}{7}$ and $b = \frac{6}{14}$ are plotted in Figure 4.6(c) for various levels of correlation. Exceptionally, correlated temperature-dependent phase noise causes the generation of entanglement beyond the initial measure and is followed by a residual entanglement at the original value. This interesting feature is exhibited by a wide range of initial states, as is depicted in Figure 4.7 as a function of a with b fixed at its maximum value $b = \sqrt{a(1-a)}$.

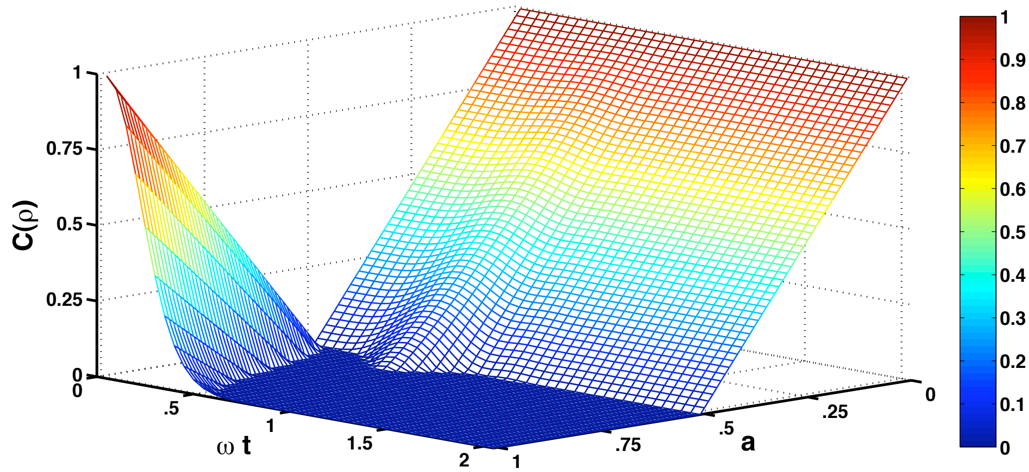

Figure 4.7: For fully correlated low-temperature noise, $\kappa = 1$, the entanglement dynamics are revealed for the spectrum of initial states in the general form matrix (4.44) as a function of parameter a. The class of entangled states described by $a < \frac{1}{2}$ all have the remarkable ability to generate entanglement beyond its initial value.

Quite remarkably, even entangled qubits that undergo ESD can have a small regeneration of entanglement when the thermal environments are fully correlated. The generation of entanglement is often a delicate feature that arises due to memory effects and quickly oscillates to zero. Having a deep understanding of what causes this regeneration would be valuable insight into the ability to control and generate entanglement.

From the current analysis, entanglement generation has strictly been a property of a complex bath correlation function of the type $\alpha(t, s) = \alpha_R(t, s) + i\alpha_I(t, s)$. For this particular model, the value of the off-diagonal elements of the reduced density operator governed by parameter b are what controlled the magnitude of the revival peak. When $b = 0$, there is no generation of entanglement, whereas when b is at the maximum value, $b = \sqrt{a(1-a)}$, the peak is at its highest. Accordingly, analyzing elements ρ_{12}, ρ_{13}, ρ_{24}, and ρ_{34} will reveal much information about this special generation feature.

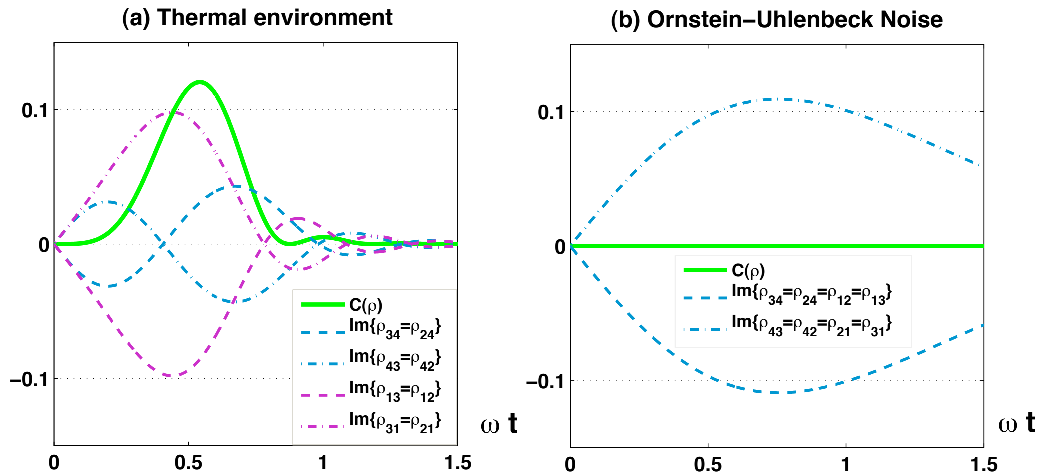

Figure 4.8: The generation of entanglement for a separable state is a feature of a complex bath, such as thermal noise, caused by the fluctuating imaginary part of the coherence.

Generally, the density matrix elements, ρ_{nm}, represent the amount of the ensemble that is in each possible qubit scenario,

$$\rho = \sum_{n,m} \rho_{nm} |n\rangle \langle m| , \tag{4.45}$$

where the indices n and m run through all possible basis elements $\{|++\rangle , |+-\rangle , |-+\rangle , |--\rangle\}$. Elements ρ_{nn} then represent the population of each basis state, leaving ρ_{nm} to describe the amount of the ensemble in various combinations of superposition qubit states. Specifically, ρ_{12} is the amount of the ensemble in state $|++\rangle \langle +-|$, which represents the state $(|+\rangle \langle +|)_A$ for qubit A and $(|+\rangle \langle -|)_B$ for qubit B. A physical interpretation of ρ_{12} is then the potential for qubit A to be in the state $|+\rangle_A$ while qubit B remains in a superposition state. Accordingly, all off-diagonal elements in the X-form matrix (4.44) with parameter b represent the amount of the ensemble that has one qubit in a specific state and the other qubit in a superposition state. It is the imaginary part of these density matrix elements that determine whether the qubits are able to generate entanglement or not and is therefore only tapped into from a complex bath correlation function, as shown in Figure 4.8. Here, the evolution of the imaginary part of the elements ρ_{12}, ρ_{13}, ρ_{24}, ρ_{34}, and their conjugates are compared between a thermal environment and Ornstein-Uhlenbeck noise for the separable state $a = b = \frac{1}{2}$.

For the temperature dependent environment, the generation of entanglement from a separable state is due to the fast fluctuation and crossings of the imaginary part of the emphasized density matrix elements. Under Ornstein-Uhlenbeck noise, no such generation is possible for the qubits and the complex terms slowly and asymptotically approach zero without ever crossing. Taking a closer look at term $\rho_{12} \left|{+}{+}\right\rangle \left\langle{+}{-}\right|$ of the ensemble state in (4.45) once again, $(\left|+\right\rangle \left\langle+\right|)_A$ can be regarded as the population of the $\left|+\right\rangle_A$ state and must contribute to the real part of ρ_{12}. In contrast, $(\left|+\right\rangle \left\langle-\right|)_B$ governs the coherence of the superposition state of qubit B and therefore must contain the complex part of ρ_{12}. In conclusion, it is asserted that the generation of entanglement between qubits interacting with correlated Ohmic phase noise is due to the fast fluctuation of single-qubit imaginary coherence. This phenomena will be studied in depth in future work.

4.3 Summary

The model of correlated noise is designed to behave as separate local baths when the environments are uncorrelated and a common bath to the qubits when the environments are fully correlated. Physically, correlated noise depends on the structure of the cavity as well as the qubit's interaction with its modes. Symmetry usually plays a very important role in the effectiveness of most entanglement schemes. Ideally, if the cavities of the distant qubits can be made identically, such as the same number of modes at equivalent frequencies and the same mean number of photons in each mode, then the local noises will become correlated and allow for the enhancement and sometimes full preservation of entanglement.

The effects of correlated noise on the entanglement dynamics of the qubits were analyzed after deriving the convolutionless master equation for non-Markovian dephasing environments, specifically Ornstein-Uhlenbeck noise and a temperature-dependent Ohmic bath. Overall, correlated non-Markovian noise enhanced a wide range of initially entangled qubit states, making correlated environments an optimal scenario for applications in quantum information science. In addition, both Ornstein-Uhlenbeck noise and temperature-dependent noise showed the ability to preserve entanglement indefinitely for the initial states in the special X-form matrix (3.16) having $a < \frac{1}{2}$, including the Bell state $|\Phi^+\rangle$. Remarkably, qubits in these states interacting with the thermal correlated environments experienced the generation of entanglement beyond the initial value, even for an initially separable state. It was demonstrated that this effect arises only for a complex bath correlation function, affecting the coherence terms of the reduced density matrix. It was concluded that the generation of qubit entanglement in this specific model of correlated noise is due to "imaginary coherence fluctuations."

Future work on this topic includes the analysis of correlated amplitude noise. Even though the dissipative Markov case in Chapter 3 showed to be unreceptive to correlated noise, there are many interesting features that arise from two qubits coupled to a common non-Markovian bath that will make correlated amplitude noise highly effective. However, as will be demonstrated in the following chapter, these facets only come out of the exact solution of the model. Because in the dissipative model the

Lindblad operator does not commute with the system Hamiltonian, $[\hat{H}_{Sys}, L] \neq 0$, the $\mathcal{O}$-operator ansatz becomes extremely difficult to develop, making an exact solution to the problem hard to find. The problem has been solved exactly for the first time in the following chapter.

Chapter 5

Fast Tracking of Entanglement via Quantum Trajectories

In the previous Chapters, it has been established that many important realizations in quantum information, such as quantum computing, quantum communication and quantum cryptography, rely on the control and generation of entanglement [1]. However, the true question arises in how to measure or compute the entanglement of a quantum system in order to effectively use that information in application. For a quantum open system described by a reduced density matrix, most definitions of entanglement pertain to a property of an ensemble, such as entanglement of formation [13, 15], $E(\rho)$, and concurrence [16], $C(\rho)$, however for possible applications in quantum information processing, the preparation of and measurement on a mixed entangled state would be quite cumbersome. A more approachable method would be to take advantage of the statistical nature of the quantum system and average over many realizations of a single system in order to infer information about the entanglement of the ensemble. Recently, entanglement unravellings in the Markov regime have been proposed [87, 88, 89]. For a general non-Markovian quantum open system, such a pure state approach is particularly useful for the numerical simulation of the tracking of entanglement information, which is known to be a hard problem due to the lack of a computable entanglement measure and a viable non-Markovian master equation [90, 91, 92, 93, 94].

In this chapter, our research serves as a first example of the efficient estimation of entanglement evolution in non-Markovian regimes without using the system density matrix. We derive the exact quantum state diffusion (QSD) equation for a pure state to estimate the entanglement evolution of a two-qubit system coupled to a bosonic heat bath at zero temperature [52, 56, 95, 96]. For a general multi-qubit system, employing quantum trajectories over density matrices becomes enormously advantageous in terms of computational exhaustion. Here, the fundamental model of two qubits interacting with a common non-Markovian environment, as depicted in Fig. 5.1, is extensively analyzed by solving the non-Markovian Quantum State Diffusion equation for exact quantum trajectories. As will be shown in the results of this work, the entanglement computed from the trajectories, $E(\psi)$, generally provides useful information about the status of the actual entanglement described by the system density matrix, $E(\rho)$. For some initial states, the trajectory entanglement gives an almost identical estimation of the system entanglement. This problem is especially relevant for superconducting qubits, which are known to be strongly coupled to their dense environments [23, 22, 21].

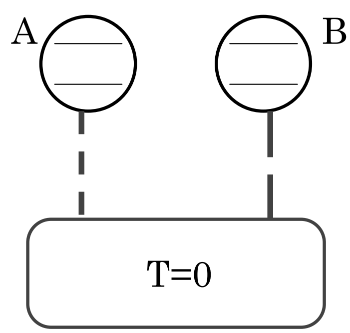

Figure 5.1: Model System 4

5.1 Deriving the QSD Equation

The model representing the most fundamental setup of a quantum computing scheme consists of two uncoupled qubits with respective transition frequencies ω_A and ω_B, which are coupled to a common zero-temperature heat bath via the interaction Hamiltonian H_{Int} in the following total Hamiltonian:

$$\begin{aligned} H_{Tot} &= H_{Sys} + H_{Bath} + H_{Int} && (5.1)\\ H_{Sys} &= \frac{\omega_A}{2}\sigma_z^A + \frac{\omega_B}{2}\sigma_z^B && (5.2)\\ H_{Bath} &= \sum_\lambda \omega_\lambda a_\lambda^\dagger a_\lambda && (5.3)\\ H_{Int} &= \sum_\lambda \left(g_\lambda^* L_A a_\lambda^\dagger + g_\lambda L_A^\dagger a_\lambda\right) + \sum_\lambda \left(f_\lambda^* L_B a_\lambda^\dagger + f_\lambda L_B^\dagger a_\lambda\right) && (5.4) \end{aligned}$$

where L_A and L_B are Lindblad operators describing the interaction of the qubits A and B with the heat bath, respectively. To explore the effect of asymmetry on the entanglement of the qubits, suppose that the coupling constants of the qubits to the bath differ by a scalar multiple, $f_\lambda = \kappa g_\lambda$, making κ a control parameter that describes the ratio of the qubits' coupling strengths, $(0 \leq \kappa \leq 1)$. It follows that the total Lindblad operator of the system is defined to be $\mathcal{L} \equiv L_A + \kappa L_B$ such that the time-dependent Schrödinger equation in the interaction picture of the bath is:

$$\frac{\partial \left|\Psi_{Tot}\right\rangle}{\partial t} = -iH_{sys}\left|\Psi_{Tot}\right\rangle - i\mathcal{L}\sum_\lambda g_\lambda^* a_\lambda^\dagger e^{i\omega_\lambda t}\left|\Psi_{Tot}\right\rangle - \mathcal{L}^\dagger \sum_\lambda g_\lambda a_\lambda e^{-i\omega_\lambda t}\left|\Psi_{Tot}\right\rangle \quad (5.5)$$

where $\left|\Psi_{Tot}\right\rangle$ represents the state vector for of the entire open quantum system.

Because the bath is at zero temperature, it can be inferred that $\left|\Psi_{Tot}\right\rangle$ is the tensor product of the system wavefunction and the environmental state

$$\left|\Psi_{Tot}\right\rangle = \left|\psi_S\right\rangle \otimes \left|z\right\rangle, \quad (5.6)$$

where $\left|z\right\rangle = \left|z_1\right\rangle \otimes \left|z_2\right\rangle \otimes \cdots \otimes \left|z_\lambda\right\rangle \otimes \cdots$ is the total environment state for which all modes are initially in the vacuum state, $\left|z_\lambda\right\rangle = \left|0_\lambda\right\rangle$, and $\left|z_\lambda\right\rangle$ is a Bragmann coherent state. Under this representation, the annihilation and creation operators in Eq. (5.5) can act on their appropriate environment eigenstates,

$$a_\lambda |z_\lambda\rangle = z_\lambda |z_\lambda\rangle \tag{5.7}$$

$$a_\lambda^\dagger |z_\lambda\rangle = \frac{\partial}{\partial z_\lambda} |z_\lambda\rangle \tag{5.8}$$

where $|z_\lambda|^2$ is the mean number of photons per mode. By then projecting Eq. (5.5) onto the total environment state, the resulting differential equation for the quantum state of the qubits, $|\psi_S\rangle = \langle z|\Psi_{Tot}\rangle$, accounts for their interaction with every mode of the environment:

$$\frac{\partial |\psi_S\rangle}{\partial t} = -iH_{Sys} |\psi_S\rangle - i\mathcal{L} \sum_\lambda g_\lambda z_\lambda e^{i\omega_\lambda t} |\psi_S\rangle - i\mathcal{L}^\dagger \sum_\lambda g_\lambda^* e^{-i\omega_\lambda t} \frac{\partial |\psi_S\rangle}{\partial z_\lambda} \tag{5.9}$$

For a large number of environmental modes, this equation is extremely difficult to solve exactly. It instead becomes convenient to model the collective effect of the environment as a random Gaussian noise variable defined as

$$z_t \equiv -i \sum_\lambda g_\lambda z_\lambda e^{i\omega_\lambda t}, \tag{5.10}$$

which has a mean value equal to zero, $\langle z_t\rangle = 0$, a variance equal to one, $\langle z_t^2\rangle - \langle z_t\rangle^2 = 1$, and is a manifestation of how the qubits couple to each mode and the mean number of photons per mode.

Inserting the noise variable into Eq. (5.9) will transform it into a stochastic differential equation where the qubit wavefunction will now depend explicitly on the noise, $|\psi_S\rangle = |\psi_t(z)\rangle$ such that

$$\frac{\partial |\psi_t(z)\rangle}{\partial z_\lambda} = \int_0^t ds \left(\frac{\delta |\psi_t(z)\rangle}{\delta z_s}\right) \left(\frac{\partial z_s}{\partial z_\lambda}\right). \tag{5.11}$$

The third term in Eq. (5.9) will then be expressed as

$$-i\sum_{\lambda} g_{\lambda}^{*} e^{-i\omega_{\lambda}t}\frac{\partial|\psi\rangle}{\partial z_{\lambda}} = -\int_0^t ds\left(|g_{\lambda}|^2 e^{-i\omega_{\lambda}(t-s)}\right)\frac{\delta|\psi\rangle}{\delta z_s} \tag{5.12}$$

$$= -\int_0^t ds M[z_t^* z_s]\frac{\delta|\psi\rangle}{\delta z_s} \tag{5.13}$$

where the bath correlation function has been derived as[1]

$$M[z_t^* z_s] = \int \frac{d^2 z}{\pi} e^{-|z|^2} z_t^* z_s = \sum_{\lambda} |g_{\lambda}|^2 e^{-i\omega_{\lambda}(t-s)} \tag{5.14}$$

due to the identity in the Bargmann coherent state representation,

$$\int_0^t ds \frac{d^2 z_{\lambda}}{\pi} e^{-|z_{\lambda}|^2} z_{\lambda}^* z_{\lambda} = 1. \tag{5.15}$$

The formal linear QSD equation [52] describing the stochastic dynamics of the quantum state of the qubits, $|\psi_t\rangle = |\psi_t(z)\rangle$, is therefore given by

$$\frac{\partial|\psi_t\rangle}{\partial t} = -iH_{Sys}|\psi_t\rangle + \mathcal{L}z_t|\psi_t\rangle - \mathcal{L}^{\dagger}\int_0^t ds M[z_t^* z_s]\frac{\delta|\psi_t\rangle}{\delta z_s}. \tag{5.16}$$

Central to the application of the QSD equation is to replace the functional derivative with a time-local operator, termed as the O-operator, such that

$$\frac{\delta|\psi_t\rangle}{\delta z_s} = \hat{\mathcal{O}}(t,s,z)|\psi_t\rangle, \tag{5.17}$$

with initial condition $\hat{\mathcal{O}}(t=s,s,z) = \mathcal{L}$. In principle, the existence of the O-operator can be seen from the stochastic propagator, $|\psi_t(z)\rangle = G(t,z)|\psi_0\rangle$ (See Appendix A), but in practice it is quite difficult to find the explicit O-operator. For the specific

[1] The notation d^2z and $|z|^2$ has been introduced in Chapter 4, Section 4.1.

two-qubit model presented here, an exact equation for the O-operator is derived upon satisfying the consistency condition [52]:

$$\frac{\delta}{\delta z_s}\left(\frac{\partial \left|\psi_t\right\rangle}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\delta \left|\psi_t\right\rangle}{\delta z_s}\right). \tag{5.18}$$

For this particular model, the consistency condition is expressed as

$$\frac{\delta \hat{O}(t,s,z)}{\delta t} = \left[-iH_{sys} + \mathcal{L}z_t - \mathcal{L}^{\dagger}\overline{O}(t,z)\right] - \mathcal{L}^{\dagger}\frac{\delta \overline{O}(t,z)}{\delta z_s} \tag{5.19}$$

where $\overline{O}(t,z) = \int_0^t ds\alpha(t-s)\hat{O}(t,s,z)$. With the initial condition, it ensures that $|\psi_t\rangle$ is a single-valued function and thus establishes a solvable QSD equation.

The solution to the QSD equation (5.16), $|\psi_t\rangle$, gives a single trajectory of the qubit state in reaction to a particular realization of the noise. Upon averaging the probability density over all possible quantum trajectories, the reduced density matrix of the qubit system is recovered, $\rho_t = M[|\psi_t\rangle \langle\psi_t|]$, and the actual entanglement evolution of the system can be calculated, $E(\rho)$. However, in experimental practices, the preparation and measurement of an ensemble of qubits becomes extremely cumbersome, especially for the eventual application in quantum computing where scalability becomes an issue. Instead, it would be extremely beneficial to have a single quantum system that evolves according to the stochastic differential equation (5.16), allowing one to calculate a single entanglement trajectory via the Concurrence [16]

$$C(\psi_t) = |\langle\psi_t|\sigma_y^A \otimes \sigma_y^B|\psi_t^*\rangle|. \tag{5.20}$$

After a significant number of realizations are produced, taking the mean over all concurrence trajectories produces a value that is pertinent to the entanglement of the ensemble system

$$\overline{C_\psi} = M[C(\psi_t)]. \tag{5.21}$$

In this way, one can efficiently compute the approximate entanglement of a quantum open system without invoking the explicit form of the density matrix and, in practice, without need of an ensemble of qubits. The actual entanglement of the ensemble, represented by the density matrix, is calculated through the concurrence [16]

$$C(\rho) = \max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right\}, \tag{5.22}$$

where λ_i $(i = 1, 2, 3, 4)$ are the eigenvalues of the matrix $\varrho = \rho(\sigma_y^A \otimes \sigma_y^B)\rho^*(\sigma_y^A \otimes \sigma_y^B)$ in descending order. Upon direct comparison, it is clear that $\overline{C_\psi}$ cannot be less than or equal to the true entanglement $C(\rho)$ due to the concavity of the concurrence calculation [16, 39]. However, $\overline{C_\psi}$ can be used as an upper bound of the actual entanglement, such that if $\overline{C_\psi} \approx 0$ then $C(\rho) \approx 0$. In fact, as shown in the following section, $\overline{C_\psi}$ provides an almost perfect estimation of the actual entanglement for some initial states. Above all, the calculation of $\overline{C_\psi}$ is much simpler than that of $C(\rho)$, especially for systems consisting of a large number of qubits. This pronounces $\overline{C_\psi}$ to be a good indicator for the actual behavior of the entanglement and will be explored in the upcoming models.

5.2 Non-Markovian Exact Entanglement Trajectories

In this section, two causes of qubit decoherence are investigated: dissipation and pure phase relaxation of the quantum state, both great challenges to maintaining robust bipartite entanglement. In both cases, the exact time-local O-operators will be derived, allowing for the efficient solution of the QSD equation [52]. Although the linear equation in (5.16) preserves the norm of the quantum state on average, $M[\langle\psi_t|\psi_t\rangle] = 1$, there is always the potential to have computational issues with sampling and, in many cases, it requires a large number of realizations to ensure efficiency. On the other hand, by deriving a dynamical equation for the normalized state vector, $|\tilde{\psi}_t\rangle = \frac{|\psi_t\rangle}{||\psi_t||}$, the norm will be robustly preserved throughout every trajectory. The dynamics of the normalized quantum state of the qubits, commonly refered to as the nonlinear QSD equation, is presented as [52]:

$$\frac{\partial |\tilde{\psi}_t\rangle}{\partial t} = -iH_{Sys}|\tilde{\psi}_t\rangle + (\mathcal{L} - \langle\mathcal{L}\rangle_t)\tilde{z}_t|\tilde{\psi}_t\rangle - \int_0^t ds\alpha(t,s)\left[(\mathcal{L}^\dagger - \langle\mathcal{L}^\dagger\rangle_t)\hat{\mathcal{O}}(t,s,\tilde{z}) - \langle(\mathcal{L}^\dagger - \langle\mathcal{L}^\dagger\rangle_t)\hat{\mathcal{O}}(t,s,\tilde{z})\rangle_t\right]|\tilde{\psi}_t\rangle\,, \quad (5.23)$$

where $\langle A\rangle_t = \langle\tilde{\psi}_t|A|\tilde{\psi}_t\rangle$ is the quantum expectation value of operator A and the shifted noise variable is $\tilde{z}_t = z_t + \int_0^t ds\alpha^*(t,s)\langle\mathcal{L}^\dagger\rangle_s$.

In addition, the non-Markovian regime is explored by modeling the bath correlation function as an Ornstein-Uhlenbeck process such that $\alpha(t,s) = \frac{\gamma}{2}e^{-\gamma|t-s|}$. This continuous random process drifts toward a stationary long-term mean and is useful for viewing various memory effects via the parameter γ, which describes the rate at which noise that is progressing in time t becomes less and less correlated to its value at a particular past time s. As γ grows very large and the correlation time $\tau_{corr} = \frac{1}{\gamma}$ becomes very short, the transition from non-Markovian to Markovian regimes will be viewed and it will be found that certain features are lost under the Markov approximation [47, 97, 7].

5.21 Dissipative Model

A dissipative interaction, which causes the quantum state to lose energy, is denoted by the Lindblad operators $L_A = \sigma_-^A$ and $L_B = \sigma_-^B$ such that $\mathcal{L} = \sigma_-^A + \kappa\sigma_-^B$. For the very first time, an exact QSD equation for the dissipative interaction model has been established here[84]. By the consistency condition of Eq. (5.18), the exact operator $\bar{O}(t,z) \equiv \int_0^t ds\alpha(t,s)\hat{O}(t,s,z)$ is presented as

$$\bar{O}(t,z) = A(t)\sigma_-^A + B(t)\sigma_-^B + F(t)\sigma_z^A\sigma_-^B + G(t)\sigma_z^B\sigma_-^A + i\left(\int_0^t ds' P(t,s')z_{s'}\right)\sigma_-^A\sigma_-^B, \tag{5.24}$$

which is valid for an arbitrary bath correlation function. By imposing the Ornstein-Uhlenbeck bath correlation function, a set of differential equations for the coefficients of the $\bar{O}$-operator are derived (See Appendix C):

$$\begin{aligned}
d_tA(t) =& -\gamma A(t) + \frac{\gamma}{2} + i\omega_A A(t) + A(t)^2 + 2\kappa F(t)G(t) + G(t)^2 - \frac{\kappa}{2}iQ(t),\\
d_tB(t) =& -\gamma B(t) + \frac{\gamma\kappa}{2} + i\omega_B B(t) + \kappa B(t)^2 + 2F(t)G(t) + \kappa F(t)^2 - \frac{1}{2}iQ(t),\\
d_tF(t) =& -\gamma F(t) + i\omega_B F(t) + F(t)[A(t)+G(t)] + B(t)[G(t)-A(t)]\\
&+ 2\kappa B(t)F(t) - \frac{1}{2}iQ(t),\\
d_tG(t) =& -\gamma G(t) + i\omega_A G(t) + \kappa F(t)[A(t)+G(t)] + \kappa B(t)[G(t)-A(t)]\\
&+ 2A(t)G(t) - \frac{\kappa}{2}iQ(t),\\
d_tQ(t) =& -2\gamma Q(t) + i(\omega_A+\omega_B)Q(t) + 2[A(t)+\kappa B(t)]Q(t) - i\gamma[F(t)+\kappa G(t)],
\end{aligned}$$

along with the explicit solution

$$P(t,s') = -2i[F(t)+\kappa G(t)]\exp\left\{\int_{s'}^t ds[-\gamma + i\omega_A + i\omega_B + 2A(s) + 2\kappa B(s)]\right\}$$

and the initial conditions $A(0) = B(0) = F(0) = G(0) = Q(0) = 0$. It should be noted that when $\kappa = 1$ and $\omega_A = \omega_B$, $A(t) = B(t)$ and $F(t) = G(t)$, representing a highly symmetrical setup where qubit A and qubit B are interchangeable, a feature known to be optimal for entanglement. Throughout the rest of the chapter, it will be assumed that the qubits share the same transition frequency, $\omega_A = \omega_B = \omega$.

Knowledge of the exact equations for the O-operator allows one to solve the nonlinear QSD Equation for various unravelings of the time evolution of the qubits initially in the maximally entangled Bell States, $|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\{|{+}{+}\rangle \pm |{-}{-}\rangle\}$ for qubits with correlated spins and $|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\{|{+}{-}\rangle \pm |{-}{+}\rangle\}$ for qubits with anti-correlated spins. In the interest of direct comparison to the exact case, the $\bar{O}$ operator in the Post-Markov approximation to is derived as

$$\bar{O}_{PM} = (f_0(t) + i\omega f_1(t))\mathcal{L} - f_2(t)(\sigma_z^A + \kappa^2 \sigma_z^B)\mathcal{L}, \tag{5.25}$$

where

$$\begin{aligned} f_0(t) &= \int_0^t \alpha(t,s)ds, \\ f_1(t) &= \int_0^t \alpha(t,s)(t-s)ds, \\ f_2(t) &= \int_0^t \int_0^s \alpha(t,s)\alpha(s,u)(t-s)du\,ds. \end{aligned} \tag{5.26}$$

The entanglement computed from the exact mean trajectory method, $\overline{C_\psi}$ is compared to that under the Post-Markov approximation, $\overline{C_{PM}(\psi)}$, as well as the entanglement of the ensemble, $C(\rho)$, in Fig. 5.2. For both initial states, $|\Psi^+\rangle$ in (a) and $|\Phi^+\rangle$ in (b), the actual entanglement $C(\rho)$ displays the repeated revival and death of entanglement known to the two-qubit model [85, 98, 99], and solved for exactly for the first time here. These trends are exhibited by the exact mean entanglement trajectory $\overline{C_\psi}$, whereas in contrast, applying the Post-Markov approximation removes all revival features of the curve. This figure demonstrates the dependence of the theory on the initial qubit state, where $|\Phi^+\rangle$ provides a much closer approximation than $|\Psi^+\rangle$.

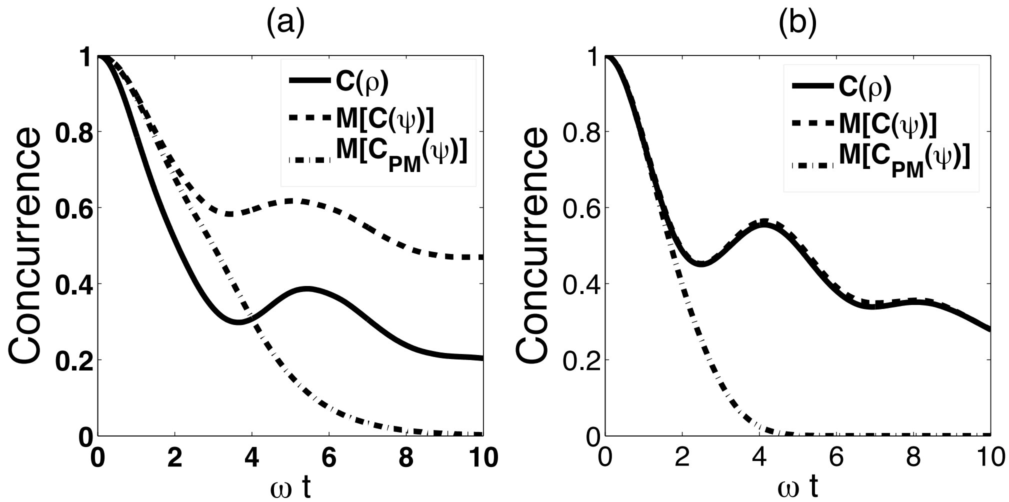

Figure 5.2: Dissipative Model: The exact $\overline{C_\psi}$ is compared to ensemble calculations in the non-Markovian regime, $C(\rho)$, and trajectory methods under the Post-Markov approximation, $\overline{C_{PM}(\psi)}$ with $\kappa = 1$ and $\gamma = 0.3$ for (a) $|\psi_0\rangle = |\Psi^+\rangle$ and (b) $|\psi_0\rangle = |\Phi^+\rangle$

However in any case, $\overline{C_\psi}$ acts as an upperbound for the exact entanglement $C(\rho)$, giving valuable information about the general trends of the entanglement evolution, such as the regeneration of entanglement and entanglement sudden death. Upon taking a closer look at $\overline{C_\psi}$ for various levels of coupling strengths and correlation times in Fig. 5.3 and Fig. 5.4, many interesting attributes of this model are revealed and the optimal conditions for entanglement are discussed.

For initial state $|\psi_0\rangle = |\Psi^+\rangle$, Fig. 5.3(a) highlights the significant revival feature of the equal couplings case, $\kappa = 1$, which shrinks as the coupling strength of qubit B is decreased. However, for this model, the asymmetry of the coupling constants causes the entanglement to decay at a much slower rate and also maintains the qubits in a higher level of entanglement for a significant period of time. This is also due to the memory effects of the non-Markovian environment with $\gamma = 0.3$, which generally allows the entanglement to remain non-zero for an extended time. In Fig. 5.3(b), the mean entanglement trajectory is compared over various correlation times for the case

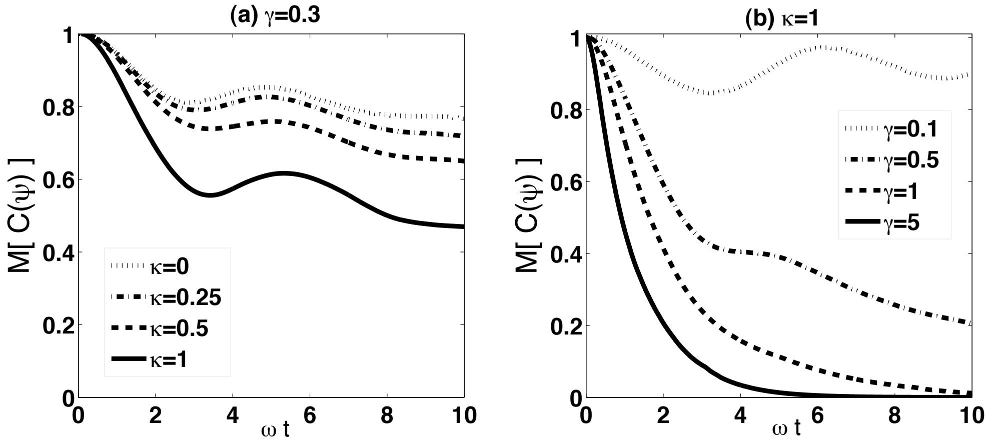

Figure 5.3: Dissipative Model: For $|\psi_0\rangle = |\Psi^+\rangle$, $\overline{C_\psi}$ over 1000 realizations is compared over (a) various values of κ for fixed $\gamma = 0.3$ and (b) various values of γ for fixed $\kappa = 1$.

of qubits with symmetrical coupling, capturing the transition from non-Markovian to Markovian regimes as γ becomes large. It is clearly shown that the revival peak of the entanglement grows as we tend toward non-Markovian conditions and eventually oscillates very close to an entangled state that will not decay, allowing one to maintain a highly entangled state over a long period of time when large memory effects are present. The importance of non-Markovian environments becomes apparent in comparison to the Markovian case, $\gamma = 5$, where the qubits are plagued with a swift decoherence and no chance for rebirth.

In Fig. 5.4(a), long time entanglment evolution from initial state $|\Phi^+\rangle$ is illustrated for various values of κ and fixed $\gamma = 0.3$, where many revival peaks are witnessed. Similar to the previous case of initial state $|\Psi^+\rangle$, tall revival peaks are displayed for symmetrical couplings, however they come at the expense of a faster disentanglement. Once again, the $\kappa = 0$ case reveals a much slower entanglement decay and remains non-zero even for long times. Comparing the effects of memory on the entanglement dynamics, Fig. 5.4(b) again demonstrates that a very large correlation time allows the quantum state to remain highly entangled for extended times.

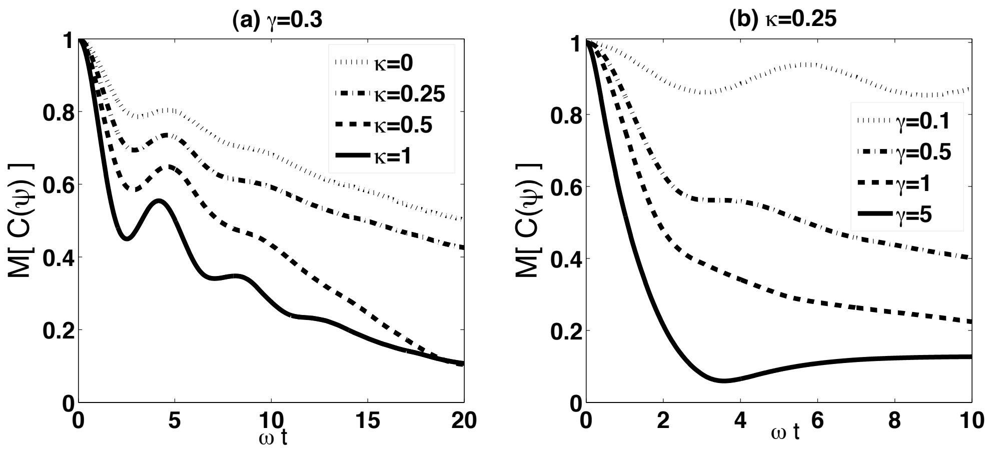

Figure 5.4: Dissipative Model: For $|\psi_0\rangle = |\Phi^+\rangle$, $\overline{C_\psi}$ is compared over (a) various values of κ for fixed $\gamma = 0.3$ for long times and (b) various values of γ for fixed $\kappa = 0.25$.

An interesting difference for this initial state is that even for fairly large $\gamma = 5$, the rebirth of entanglement is still a dominant feature.

5.3 Dephasing Model

As another important case, a dephasing type of interaction is considered, which provides an example of pure decoherence without dissipation. Described by the two Lindblad operators $L_A = \sigma_z^A$ and $L_B = \sigma_z^B$, the consistency conditions of Eq. (5.18) result in the noise-free exact O-operator $\hat{\mathcal{O}}(t,s) = \mathcal{L} = \sigma_z^A + \kappa\sigma_z^B$ due to $[H_{\text{sys}}, \mathcal{L}] = 0$ and $\mathcal{L}^\dagger = \mathcal{L}$. Applying the Ornstein-Uhlenbeck bath correlation function then results in the noise-independent time-local operator $\bar{\mathcal{O}}(t) = \frac{1}{2}(1 - e^{\gamma|t|})(\sigma_z^A + \kappa\sigma_z^B)$ that facilitates a solution to the exact non-Markovian QSD equation. The mean entanglement trajectories for the dephasing model are plotted in Figs. (5.5) and (5.6).

In Fig. 5.5(a), the dotted curve, $\kappa = 0$, represents the scenario of qubit A interacting with the heat bath while qubit B is a free particle. As is expected of the single-qubit dephasing channel [37, 38], the entanglement of the qubits asymptotically

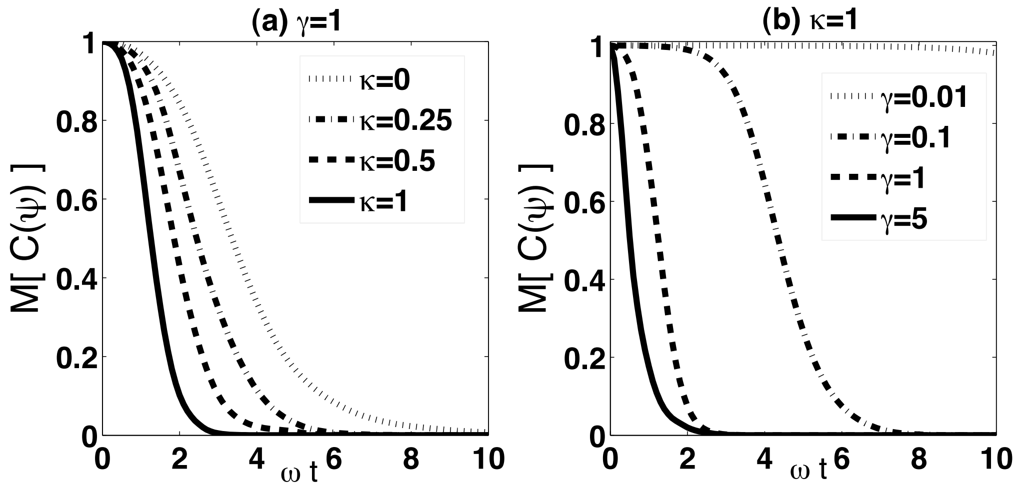

Figure 5.5: Dephasing Model: For $|\psi_0\rangle = |\Psi^{\pm}\rangle$, $\overline{C_\psi}$ over 1000 realizations is compared over (a) various values of κ for fixed $\gamma = 1$ and (b) various values of γ for fixed $\kappa = 1$.

decays to zero. Moreover, as we introduce the interaction of qubit B to the environment through $\kappa \neq 0$, the disentanglement rate between the qubits only increases and causes a faster death of entanglement. In Fig. 5.5(b), the very non-Markovian case, $\gamma = 0.01$, where the memory of the system extends much further into the past, reveals the preservation of entanglement for a considerable length of time before beginning to decay. In the limit as γ approaches 0 one would expect entanglement to be sustained at the maximum value eternally. As the memory of the system is shortened into the Markov regime, the entanglement curves reveal a steeper and steeper descent toward zero entanglement.

The same analysis was applied to qubits with initial state $|\Phi^{\pm}\rangle$ and shown in Fig. 5.6. In Fig. 5.6(a) it immediately becomes apparent that when the coupling constants of the qubits to the heat bath are equal, $\kappa = 1$, then the initially entangled state is protected due to the symmetry between the two qubits. When the qubits are not coupled to the modes of the heat bath in exactly the same way, $\kappa \neq 1$, the entanglement will eventually decay to zero. Similar to the dissipative model,

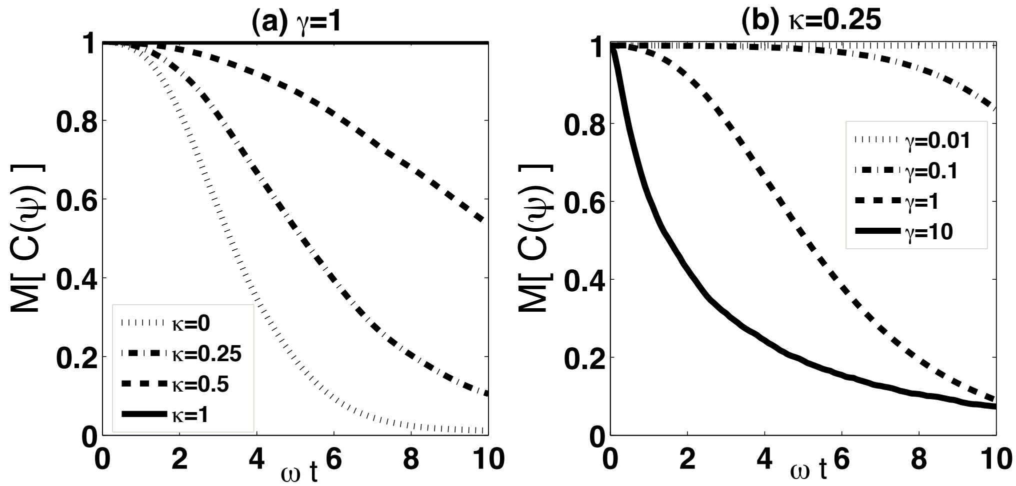

Figure 5.6: Dephasing Model: For $|\psi_0\rangle = |\Phi^{\pm}\rangle$, $\overline{C_\psi}$ is compared over (a) various values of κ for fixed $\gamma = 1$ and (b) various values of γ for fixed $\kappa = 0.25$.

Fig. 5.6(b) displays the prolonged entanglement of the qubits in the non-Markovian case, $\gamma = 0.01$, and the faster disentanglement rate of the Markov approximation, $\gamma = 10$.

5.4 Summary

In conclusion, the dynamical entanglement of a non-Markovian open system can be efficiently estimated by employing exact quantum diffusive trajectories. In particular, it has been shown that the entanglement dynamics of the system are very sensitive to which initial state the qubits evolved from, how the qubits are coupled to the heat bath, and the correlation time of the environment. It is emphasized here that under the Markov approximation, the entanglement for both sets of Bell states was characterized by fast disentanglement and suppressed revival features, whereas in the non-Markovian regime, large revivals were witnessed and for an extensively long correlation time, the qubits remained nearly maximally entangled for long times. It was demonstrated that the optimal conditions for maintaining a high level of entanglement for long periods of time are the symmetric coupling strengths of the qubits to the environment and for the autocorrelation time of the environment to be very long. This again emphasizes the importance of memory effects on the dynamics of an open quantum system and the role of symmetry on entanglement preservation.

Chapter 6

General Conclusions and Future Work

Although the threat of environmental noise to bipartite entanglement is imminent in open quantum systems, the research presented here sheds light on the possibility of overcoming decoherence and preserving entanglement long enough to be used in application. The theoretical approaches taken here present the exact entanglement evolution of fundamental qubit systems, some derived exactly for the first time, revealing the limits of certain scenarios and the most optimal situations for preserving entanglement. In all models an optimistic future for the preservation of entanglement was revealed.

The fundamental two-qubit local noise model, representing a basic quantum communication scheme, is well-known to be very vulnerable to a fast disentanglement, especially since local operations can in no way enhance the entanglement of the qubits. Instead, by augmenting a global feature of the total system, one can effectively modulate the future evolution of the entanglement. Recalling that the only bipartite system known to naturally provide the preservation of entanglement for certain initial states is one where the dephasing environment is common to both qubits, it is worthwhile to transform the local noise model into one which mimics the mutual environment. Because of the nonlocality of entanglement by definition, providing identical local environmental conditions is enough for the qubits to perceive themselves to be in a common bath, due to the statistical correlation which arises between the noise variables, therefore allowing for the modulation and even preservation of entanglement.

In the semi-classical model of Chapter 3, the preliminary study in the Markov regime revealed a great promise for the preservation of a wide range of initial entangled states via correlated phase noise. This model was explored in much more depth in the fully quantized model of Chapter 4, additionally in the non-Markov regime. The exact formulation of the QSD equation allowed for the construction of correlated phase noise from first principles. It revealed the necessary conditions for the local noises to be considered indeed fully correlated: (1) the environments must have the same number of modes at the same eigenfrequencies, (2) the mean number of photons per mode must be identical for both environments, (3) the qubits must couple with the same strength to each mode. Although these conditions might seem quite restrictive, even a very loose correlation between the local phase noises provides an enhancement of the bipartite entanglement, making correlated noise quite generally a good recipe for improvement, especially for non-Markovian environments. The positive effects of correlated phase noise were revealed for Ornstein-Uhlenbeck noise, where the same class of initial states was preserved indefinitely. Most remarkably, a thermal environment presented the ability to generate entanglement beyond its initial value and stabilize at a finite measure.

Although the results were quite positive for overcoming decoherence due to phase relaxation, the preliminary study of correlated amplitude noise in the semi-classical case revealed much different results. In general, dissipation is known to be very detrimental to bipartite entanglement, which most commonly ends up disentangling in a finite time. In addition, the fast descent to zero entanglement known to the Markov regime is often quite rigid and hard to overcome. It is then not quite surprising that mimicking a common dissipative Markov environment with correlated noise would not necessary improve the bipartite entanglement conditions. It in turn caused the measure of entanglement to be reduced to less than the value it would have with plain statistically independent classical noise. However, it is very likely that this is only true in the Markov regime, as a common non-Markovian dissipative bath was shown to have many entanglement revivals, which can essentially be tapped into via correlated noise. The exact derivation of the qubit dynamics under dissipation is quite difficult to resolve, as was shown in Chapter 5. However with the formalism already derived, future work dedicated to solving the two qubit correlated non-Markovian amplitude noise model will undoubtedly reveal interesting results.

As was discussed in the context of correlated noise, another very fundamental model studied in Chapter 5 was two qubits in the presence of a common non-Markovian environment. This scheme is quite generally important for the fact that qubits will be placed within certain devices, which all have very dense environments, and fall victim to decoherence and disentanglement. Solving this model exactly for the first time in the dephasing and dissipative cases shed much light on what conditions are optimal for maintaining a high level of entanglement for such a system.

First, it was revealed that symmetry plays a very important role in the preservation of entanglement. For instance, in the dephasing case, entanglement was completely preserved for the initial Bell state $|\Phi^{\pm}\rangle$ only when the qubits coupled with equal strength to the modes of the environment. Any difference in coupling strengths resulted in the eventual death of entanglement, even if very slow. In the dissipative case, symmetrical coupling constants also provided a much slower decay of entanglement for the two qubits, however never with the prospect for a full robust sustenance.

Second, by modeling the noise as an Ornstein-Uhlenbeck process, various effects of the memory of the environment were explored. The bath correlation function quantifies how the noise at present time depends on its value at a particular time in the past, demonstrating the strength of the environment's "memory." In the case of Ornstein-Uhelenbeck noise, the correlation exponentially decays as the noise progresses further and further ahead. The decay rate can be very slow, reflecting very non-Markovian systems where memory effects are strong, or a very rapid decay rate reminiscent of the Markov regime. In all cases studied, both phase relaxation and dissipation, all initially entangled states disentangled at a significantly slower rate when the environment had a strong memory of previous times. In this way, the non-Markovian regime becomes very important to study for multipartite systems for applications in quantum information processing.

Lastly, the fast-tracking of entanglement was discussed for the first time with respect to non-Markovian systems. Because, by definition, entanglement is a property of the ensemble state of the qubits rather than a single trajectory, the discussion about taking real-time measurements of entanglement on a non-Markovian system becomes quite complicated. Instead, it was proposed in Chapter 5 that calculating the entanglement evolution of individual trajectories of the qubits, which is stochastic due to fluctuations from the noise, and then averaging over many realizations can provide a reasonably good indication of the general trends of the actual entanglement. Because of the way entanglement is calculated via the concurrence, it is known that these mean entanglement trajectories could never exactly equal the actual entanglement but rather provides an upperbound. For the exact two-qubit model with a common dissipative environment, the mean entanglement trajectories properly reproduced all trends of the actual entanglement, such as the peaks and troughs of regeneration, and proved to be a much better approximation than the commonly used Post-Markov approximation. In fact, for the initial state $|\Phi^{\pm}\rangle$, the mean entanglement trajectories predicted an almost identical evolution of the exact entanglement, making it extremely useful for future approximate calculations of very complex multipartite systems.

In conclusion, there is great optimism for the robust generation and preservation of entanglement for qubits in an open quantum system. While the study presented in this thesis extended only to a bipartite system, the study of more complicated networks of qubits will reveal the limits of the many proposed applications of quantum information science and potentially will reveal new aspects of multipartite theory that is unimmaginable. By continuing to study these complex fundamental systems, we inch closer to realizing the rich applications that arose from quantum entanglement.

Appendix A

Existence of the O-operator

Because of the nonlocal behavior of the general QSD equation proposed by Diosi, Strunz, and Gisin [52], it becomes essential to replace the functional derivative of the wavefunction with respect to noise with a linear time-local operator, termed the $\mathcal{O}$-operator, in order to obtain an exact solution. The existence of the O-operator can be seen from the linear propagator $\hat{G}$ for a general QSD equation, where $\psi_t(z) = \hat{G}(t, x^*)\psi_0$ [100]. Consequently,

$$\begin{aligned}
\frac{\delta\psi_t(x^*)}{\delta x_s^*} &= \left[\frac{\delta}{\delta x_s^*}\hat{G}(t, x^*)\right]\psi_0 \\
&= \left(\frac{\delta}{\delta x_s^*}\hat{G}\right)\hat{G}^{-1}\psi_t(x^*) \\
&= \hat{\mathcal{O}}(t, s, x^*)\psi_t(x^*).
\end{aligned}$$

At the moment, it is still unclear under what mathematical conditions an exact $\mathcal{O}$-operator can be determined, however it is known that perturbative $\mathcal{O}$-operators can always be obtained [56].

In general, it is always possible to find an approximate $\mathcal{O}$-operator, for example in the Post-Markov and Markov approximation. However, the difficult task is deriving the exact $\mathcal{O}$-operator, which, for many models, does not always have an explicit form. Many physically interesting models have employed the O-operator approach [56, 101, 46, 102, 68, 65, 103] and up until now, the exact $\mathcal{O}$-operator has been established only in the following cases: one qubit in a dephasing and dissipative environment [52, 56, 101], one harmonic oscillator in Brownian motion and dissipative environment [52, 56, 46, 102], one harmonic oscillator in Brownian motion with finite temperature [68], a cavity mode in a dissipative environment with zero and finite temperature [65], and one three-level atom in a dissipative environment [103]. For the first time, the exact $\mathcal{O}$-operators have been obtained for the two qubit models, first in the presence of non-Markovian local correlated environments (Chapter 4) and then for the two-qubit model with a common non-Markovian dissipative and dephasing environment (Chapter 5).

Appendix B

Dynamic equations for the exact $\mathcal{O}$-operators: Correlated Noise Model

In this Appendix, the dynamic equations governing the exact formulation of the $\mathcal{O}$-operators will be derived in full for the specific model of correlated quantized phase noise.

In Chapter 4, the QSD equation (4.22) describing two qubits experiencing phase relaxation due to their interaction with local noisy environments, x_t and y_t, which are statistically correlated, was presented as

$$\frac{\partial |\psi_t\rangle}{\partial t} = -i\hat{H}_{Sys}|\psi_t\rangle + \sigma_z^A x_t |\psi_t\rangle - \sigma_z^A \int_0^t ds \left\{ M[x_t^* x_s] \frac{\delta |\psi_t\rangle}{\delta x_s} + M[x_t^* y_s] \frac{\delta |\psi_t\rangle}{\delta y_s} \right\}$$
$$+ \sigma_z^B y_t |\psi_t\rangle - \sigma_z^B \int_0^t ds \left\{ M[y_t^* x_s] \frac{\delta |\psi_t\rangle}{\delta x_s} + M[y_t^* y_s] \frac{\delta |\psi_t\rangle}{\delta y_s} \right\}. \tag{6.1}$$

It is of common practice to replace the functional derivative of the state vector $|\psi_t\rangle$ with respect to the noise variables by the time-local $\mathcal{O}$-operators

$$\frac{\delta |\psi_t\rangle}{\delta x_s} = \hat{\mathcal{O}}_A(t,s,x,y)|\psi_t\rangle \tag{6.2}$$
$$\frac{\delta |\psi_t\rangle}{\delta y_s} = \hat{\mathcal{O}}_B(t,s,x,y)|\psi_t\rangle \tag{6.3}$$

which have the initial conditions

$$\hat{\mathcal{O}}_A(t=s,s,x,y) = \sigma_z^A \tag{6.4}$$
$$\hat{\mathcal{O}}_B(t=s,s,x,y) = \sigma_z^B. \tag{6.5}$$

One can derive a set of differential equations for the time dependence of the operators by applying the consistency conditions,

$$\frac{\partial}{\partial t}\left(\frac{\delta\,|\psi_t\rangle}{\delta x_s}\right) = \frac{\delta}{\delta x_s}\left(\frac{\partial\,|\psi_t\rangle}{\partial t}\right) \tag{6.6}$$

$$\frac{\partial}{\partial t}\left(\frac{\delta\,|\psi_t\rangle}{\delta y_s}\right) = \frac{\delta}{\delta y_s}\left(\frac{\partial\,|\psi_t\rangle}{\partial t}\right) \tag{6.7}$$

which refer to the ordering of the derivatives and will put restraints on the $\mathcal{O}$-operators that ensure the wavefunction $|\psi_t\rangle$ remains a single-valued function. From Eqs. (6.6) and (6.7) the following differential equations for the $\mathcal{O}$-operators can be derived:

$$\begin{aligned}\frac{\partial\hat{\mathcal{O}}_A}{\partial t} = \left[-i\hat{H}_{Sys},\hat{\mathcal{O}}_A\right] &+ \left[\sigma_z^A x_t - \sigma_z^A\int_0^t dsM[x_t^* x_s]\hat{\mathcal{O}}_A - \sigma_z^A\int_0^t dsM[x_t^* y_s]\hat{\mathcal{O}}_B,\hat{\mathcal{O}}_A\right]\\ &+ \left[\sigma_z^B y_t - \sigma_z^B\int_0^t dsM[y_t^* x_s]\hat{\mathcal{O}}_A - \sigma_z^B\int_0^t dsM[y_t^* y_s]\hat{\mathcal{O}}_B,\hat{\mathcal{O}}_A\right]\\ &- \sigma_z^A\frac{\delta}{\delta x_s}\left(\int_0^t dsM[x_t^* x_s]\hat{\mathcal{O}}_A\right) - \sigma_z^A\frac{\delta}{\delta x_s}\left(\int_0^t dsM[x_t^* y_s]\hat{\mathcal{O}}_B\right)\\ &- \sigma_z^B\frac{\delta}{\delta x_s}\left(\int_0^t dsM[y_t^* x_s]\hat{\mathcal{O}}_A\right) - \sigma_z^B\frac{\delta}{\delta x_s}\left(\int_0^t dsM[y_t^* y_s]\hat{\mathcal{O}}_B\right)\end{aligned} \tag{6.8}$$

$$\begin{aligned}\frac{\partial\hat{\mathcal{O}}_B}{\partial t} = \left[-i\hat{H}_{Sys},\hat{\mathcal{O}}_B\right] &+ \left[\sigma_z^A x_t - \sigma_z^A\int_0^t dsM[x_t^* x_s]\hat{\mathcal{O}}_A - \sigma_z^A\int_0^t dsM[x_t^* y_s]\hat{\mathcal{O}}_B,\hat{\mathcal{O}}_B\right]\\ &+ \left[\sigma_z^B y_t - \sigma_z^B\int_0^t dsM[y_t^* x_s]\hat{\mathcal{O}}_A - \sigma_z^B\int_0^t dsM[y_t^* y_s]\hat{\mathcal{O}}_B,\hat{\mathcal{O}}_B\right]\\ &- \sigma_z^A\frac{\delta}{\delta y_s}\left(\int_0^t dsM[x_t^* x_s]\hat{\mathcal{O}}_A\right) - \sigma_z^A\frac{\delta}{\delta y_s}\left(\int_0^t dsM[x_t^* y_s]\hat{\mathcal{O}}_B\right)\\ &- \sigma_z^B\frac{\delta}{\delta y_s}\left(\int_0^t dsM[y_t^* x_s]\hat{\mathcal{O}}_A\right) - \sigma_z^B\frac{\delta}{\delta y_s}\left(\int_0^t dsM[y_t^* y_s]\hat{\mathcal{O}}_B\right)\end{aligned} \tag{6.9}$$

From the initial conditions in Eqs. (6.4) and (6.5) we can assume that the $\mathcal{O}$-operators for this model are of the general form:

$$\hat{\mathcal{O}}_A(t,s,x,y) = c_1(t,s,x,y)\sigma_z^A + c_2(t,s,x,y) \tag{6.10}$$

$$\hat{\mathcal{O}}_B(t,s,x,y) = d_1(t,s,x,y)\sigma_z^B + d_2(t,s,x,y) \tag{6.11}$$

where the coefficients contain all possible dependence on time and noise of the $\mathcal{O}$-operator and initially $c_1(t,s,x,y) = d_1(t,s,x,y) = 1$ and $c_2(t,s,x,y) = d_2(t,s,x,y) = 0$. From experience, if any further operators besides the ones present in the Lindblad operator are needed in the full description of the $\mathcal{O}$-operator, they will arise from the commutators in Eqs. (6.8) and (6.9). Inserting Eqs. (6.10) and (6.11) into the differential equations for the $\mathcal{O}$-operators gives the following pair of differential equations:

$$
\begin{aligned}
\frac{\partial c_1(t,s,x,y)}{\partial t}\sigma_z^A + \frac{\partial c_2(t,s,x,y)}{\partial t}\mathbb{1} = &- \sigma_z^A \frac{\delta}{\delta x_s}\left(\int_0^t dsM[x_t^* x_s] c_1(t,s,x,y)\right)\sigma_z^A \\
-\sigma_z^A \frac{\delta}{\delta x_s}\left(\int_0^t dsM[x_t^* y_s] d_1(t,s,x,y)\right)\sigma_z^B &- \sigma_z^B \frac{\delta}{\delta x_s}\left(\int_0^t dsM[y_t^* x_s] c_1(t,s,x,y)\right)\sigma_z^A \\
-\sigma_z^B \frac{\delta}{\delta x_s}\left(\int_0^t dsM[y_t^* y_s] d_1(t,s,x,y)\right)\sigma_z^B &- \sigma_z^A \frac{\delta}{\delta x_s}\left(\int_0^t dsM[x_t^* x_s] c_2(t,s,x,y)\right) \\
-\sigma_z^A \frac{\delta}{\delta x_s}\left(\int_0^t dsM[x_t^* y_s] d_2(t,s,x,y)\right) &- \sigma_z^B \frac{\delta}{\delta x_s}\left(\int_0^t dsM[y_t^* x_s] c_2(t,s,x,y)\right) \\
&- \sigma_z^B \frac{\delta}{\delta x_s}\left(\int_0^t dsM[y_t^* y_s] d_2(t,s,x,y)\right) \\
\frac{\partial d_1(t,s,x,y)}{\partial t}\sigma_z^B + \frac{\partial d_2(t,s,x,y)}{\partial t}\mathbb{1} = &- \sigma_z^A \frac{\delta}{\delta y_s}\left(\int_0^t dsM[x_t^* x_s] c_1(t,s,x,y)\right)\sigma_z^A \\
-\sigma_z^A \frac{\delta}{\delta y_s}\left(\int_0^t dsM[x_t^* y_s] d_1(t,s,x,y)\right)\sigma_z^B &- \sigma_z^B \frac{\delta}{\delta y_s}\left(\int_0^t dsM[y_t^* x_s] c_1(t,s,x,y)\right)\sigma_z^A \\
-\sigma_z^B \frac{\delta}{\delta y_s}\left(\int_0^t dsM[y_t^* y_s] d_1(t,s,x,y)\right)\sigma_z^B &- \sigma_z^A \frac{\delta}{\delta y_s}\left(\int_0^t dsM[x_t^* x_s] c_2(t,s,x,y)\right) \\
-\sigma_z^A \frac{\delta}{\delta y_s}\left(\int_0^t dsM[x_t^* y_s] d_2(t,s,x,y)\right) &- \sigma_z^B \frac{\delta}{\delta y_s}\left(\int_0^t dsM[y_t^* x_s] c_2(t,s,x,y)\right) \\
&- \sigma_z^B \frac{\delta}{\delta y_s}\left(\int_0^t dsM[y_t^* y_s] d_2(t,s,x,y)\right)
\end{aligned}
$$

By equating all coefficients of the operators in the above equations, it is shown that:

$$\frac{\partial c_1(t,s,x,y)}{\partial t} = -\frac{\delta}{\delta x_s}\left(\int_0^t dsM[x_t^* x_s]c_2(t,s,x,y)\right) - \frac{\delta}{\delta x_s}\left(\int_0^t dsM[x_t^* y_s]d_2(t,s,x,y)\right) \tag{6.12}$$

$$\frac{\partial c_2(t,s,x,y)}{\partial t} = -\frac{\delta}{\delta x_s}\left(\int_0^t dsM[x_t^* x_s]c_1(t,s,x,y)\right) - \frac{\delta}{\delta x_s}\left(\int_0^t dsM[y_t^* y_s]d_1(t,s,x,y)\right) \tag{6.13}$$

$$\frac{\partial d_1(t,s,x,y)}{\partial t} = -\frac{\delta}{\delta y_s}\left(\int_0^t dsM[y_t^* x_s]c_2(t,s,x,y)\right) - \frac{\delta}{\delta y_s}\left(\int_0^t dsM[y_t^* y_s]d_2(t,s,x,y)\right) \tag{6.14}$$

$$\frac{\partial d_2(t,s,x,y)}{\partial t} = -\frac{\delta}{\delta y_s}\left(\int_0^t dsM[x_t^* x_s]c_1(t,s,x,y)\right) - \frac{\delta}{\delta y_s}\left(\int_0^t dsM[y_t^* y_s]d_1(t,s,x,y)\right) \tag{6.15}$$

and

$$\frac{\delta}{\delta x_s}\left(\int_0^t dsM[x_t^* y_s]d_1(t,s,x,y)\right) = \frac{\delta}{\delta x_s}\left(\int_0^t dsM[y_t^* x_s]c_1(t,s,x,y)\right) \tag{6.16}$$

$$\frac{\delta}{\delta x_s}\left(\int_0^t dsM[y_t^* x_s]c_2(t,s,x,y)\right) = \frac{\delta}{\delta x_s}\left(\int_0^t dsM[y_t^* y_s]d_2(t,s,x,y)\right) \tag{6.17}$$

$$\frac{\delta}{\delta y_s}\left(\int_0^t dsM[x_t^* y_s]d_1(t,s,x,y)\right) = \frac{\delta}{\delta y_s}\left(\int_0^t dsM[y_t^* x_s]c_1(t,s,x,y)\right) \tag{6.18}$$

$$\frac{\delta}{\delta y_s}\left(\int_0^t dsM[x_t^* x_s]c_2(t,s,x,y)\right) = \frac{\delta}{\delta y_s}\left(\int_0^t dsM[x_t^* y_s]d_2(t,s,x,y)\right). \tag{6.19}$$

From the latter four equations (6.16)-(6.19) it can be deduced:

$$\frac{\delta c_1(t,s,x,y)}{\delta x_s} = \frac{M[x_t^* y_s]}{M[y_t^* x_s]}\frac{\delta d_1(t,s,x,y)}{\delta x_s}$$
$$\frac{\delta c_2(t,s,x,y)}{\delta x_s} = \frac{M[y_t^* y_s]}{M[y_t^* x_s]}\frac{\delta d_2(t,s,x,y)}{\delta x_s}$$
$$\frac{\delta c_1(t,s,x,y)}{\delta y_s} = \frac{M[x_t^* y_s]}{M[y_t^* x_s]}\frac{\delta d_1(t,s,x,y)}{\delta y_s}$$
$$\frac{\delta c_2(t,s,x,y)}{\delta y_s} = \frac{M[x_t^* y_s]}{M[x_t^* x_s]}\frac{\delta d_2(t,s,x,y)}{\delta y_s}$$

which can be inserted into equations (6.12)-(6.15) to produce:

$$\frac{\partial c_1(t,s,x,y)}{\partial t} = -\int_0^t ds\left[\frac{M[x_t^* x_s]M[y_t^* y_s]}{M[y_t^* x_s]} + M[x_t^* y_s]\right]\frac{\delta d_2(t,s,x,y)}{\delta x_s} \quad (6.20)$$
$$\frac{\partial c_2(t,s,x,y)}{\partial t} = -\int_0^t ds\left[\frac{M[x_t^* x_s]M[x_t^* y_s]}{M[y_t^* x_s]} + M[y_t^* y_s]\right]\frac{\delta d_1(t,s,x,y)}{\delta x_s} \quad (6.21)$$
$$\frac{\partial d_1(t,s,x,y)}{\partial t} = -\int_0^t ds\left[\frac{M[y_t^* x_s]M[x_t^* y_s]}{M[x_t^* x_s]} + M[y_t^* y_s]\right]\frac{\delta d_2(t,s,x,y)}{\delta y_s} \quad (6.22)$$
$$\frac{\partial d_2(t,s,x,y)}{\partial t} = -\int_0^t ds\left[\frac{M[x_t^* x_s]M[x_t^* y_s]}{M[y_t^* x_s]} + M[y_t^* y_s]\right]\frac{\delta d_1(t,s,x,y)}{\delta y_s} \quad (6.23)$$

From Eqs. (6.22) and (6.23), it is apparent that the time-dependence of coefficients $d_1(t,s,x,y)$ and $d_2(t,s,x,y)$ are independent of the coefficients $c_1(t,s,x,y)$ and $c_2(t,s,x,y)$. Furthermore, because these coefficients have the initial conditions $d_1(t=s,s,x,y) = d_2(t=s,s,x,y) = 0$, their solution will therefore be trivial

$$d_1(t,s,x,y) = d_2(t,s,x,y) = 0.$$

Equations (6.20) and (6.21) then also become greatly simplified:

$$\frac{\partial c_1(t,s,x,y)}{\partial t} = 0 \tag{6.24}$$
$$\frac{\partial c_2(t,s,x,y)}{\partial t} = 0 \tag{6.25}$$

in which case the coefficients will remain constant in their initial conditions:

$$c_1(t,s,x,y) = c_2(t,s,x,y) = 1.$$

This concludes the derivation of the $\mathcal{O}$-operators presented in Chapter 4,

$$\hat{\mathcal{O}}_A(t,s,x,y) = \sigma_z^A, \tag{4.27}$$
$$\hat{\mathcal{O}}_B(t,s,x,y) = \sigma_z^B. \tag{4.28}$$

The results obtained from this approach quite generally apply to all dephasing models for which the Lindblad operators and the system Hamiltonian contain combinations of σ_z^A and σ_z^B operators and therefore commute, $[H_{Sys}, L] = 0$. Under those conditions, the $\mathcal{O}$-operator can simply be concluded to be equal to the Lindblad operator upon sight. For example, recall the QSD equation obtained in Chapter 4 after applying the ansatz for correlated noise:

$$\begin{aligned}\frac{\partial |\psi_t\rangle}{\partial t} = -i\hat{H}_{Sys}|\psi_t\rangle + (\sigma_z^A + \kappa\sigma_z^B)x_t|\psi_t\rangle - (\sigma_z^A + \kappa\sigma_z^B)^2 \int_0^t dsM[x_t^* x_s]|\psi_t\rangle \\ + (1-\kappa)\sigma_z^B r_t|\psi_t\rangle - (1-\kappa)^2(\sigma_z^B)^2 \int_0^t dsM[r_t^* r_s]|\psi_t\rangle\,. \end{aligned} \tag{4.32}$$

By inspection, it is found that the two Lindblad operators are now $L_1 = \sigma_z^A + \kappa\sigma_z^B$ and $L_2 = (1-\kappa)\sigma_z^B$, which obey the properties $L_{1,2}^\dagger = L_{1,2}$ and commute with the system

Hamiltonian, $[H_{Sys}, L_1] = [H_{Sys}, L_2] = 0$ as well as with either other $[L_1, L_2] = 0$. Therefore, it is known that the $\mathcal{O}$-operators will simply be equal to their respective Lindblad operators, recovering the result

$$\hat{\mathcal{O}}_C(t,s,x,r)\left|\psi_t\right\rangle = (\sigma_z^A + \kappa\sigma_z^B)\left|\psi_t\right\rangle \tag{4.38}$$

$$\hat{\mathcal{O}}_D(t,s,x,r)\left|\psi_t\right\rangle = (1-\kappa)\sigma_z^B\left|\psi_t\right\rangle . \tag{4.39}$$

It is almost always true that the $\mathcal{O}$-operator will be equal to the Lindblad operator when the Lindblad operators commute with each other as well as the system Hamiltonian, making an exact solution to the QSD and master equations quite easy to obtain. In contrast, the situation becomes much more complicated when they do not commute, as will be seen in the following appendix for the dissipative model.

Appendix C

Dynamic equations for the exact $\mathcal{O}$-operators: Common non-Markovian Environment

In this Appendix, the dynamic equations for the coefficients of the exact $\mathcal{O}$-operator will be derived for the two qubit model, in which the qubits interact asymmetrically to a common non-Markovian environment.

In Chapter 5, the QSD equation describes the dynamics of the pure quantum state ψ_t under the influence of the complex stochastic Gaussian process z_t was presented as the following [55, 52]

$$\frac{\partial |\psi_t\rangle}{\partial t} = -iH_{Sys} |\psi_t\rangle + \mathcal{L}z_t |\psi_t\rangle - \mathcal{L}^\dagger \int_0^t dsM[z_t^* z_s] \frac{\delta |\psi_t\rangle}{\delta z_s}. \tag{5.16}$$

This formal linear QSD equation becomes non-local because of the functional derivative with respect to noise, so the time-local non-Markovian QSD equation cannot be derived if the functional derivative cannot be replaced with a linear operator acting on the state vector ψ_t, such that

$$\frac{\delta |\psi_t\rangle}{\delta z_s} = \hat{\mathcal{O}}(t,s,z) |\psi_t\rangle \, . \tag{5.17}$$

By the consistency condition in Eq. (5.18), one gets the equation of motion for the O-operator:

$$\frac{\partial \hat{\mathcal{O}}(t,s,z)}{\partial t} = \quad [-iH_{Sys} + \mathcal{L}z_t - \mathcal{L}^\dagger \bar{\mathcal{O}}(t,z), \hat{\mathcal{O}}(t,s,z)] - \mathcal{L}^\dagger \frac{\delta \bar{\mathcal{O}}(t,z)}{\delta z_s}, \tag{6.26}$$

where $\bar{\mathcal{O}}(t,z) = \int_0^t ds\alpha(t,s)\hat{\mathcal{O}}(t,s,z)$ and $\alpha(t,s) = M[z_t^* z_s]$ is the bath correlation function, which will later taken to be an Ornstein-Uhlenbeck process.

For the dissipative model, where $\mathcal{L} = \sigma_-^A + \kappa\sigma_-^B$, it can be shown that the O-operator takes the following form:

$$\begin{aligned}\hat{\mathcal{O}}(t,s,z) =& a(t,s)\sigma_-^A + b(t,s)\sigma_-^B + f(t,s)\sigma_z^A\sigma_-^B + g(t,s)\sigma_-^A\sigma_z^B \\ &+ i\left(\int_0^t ds' p(t,s,s')z_{s'}\right)\sigma_-^A\sigma_-^B,\end{aligned} \tag{6.27}$$

where equations of motion for $a(t,s)$, $b(t,s)$, $f(t,s)$, $g(t,s)$, and $p(t,s,s')$ can be derived from Eq. (6.26). It then follows that by definition of $\bar{\mathcal{O}}(t,z) = \int_0^t ds\alpha(t,s)\hat{\mathcal{O}}(t,s,z)$,

$$\begin{aligned}\bar{\mathcal{O}}(t,z) =& A(t)\sigma_-^A + B(t)\sigma_-^B + F(t)\sigma_z^A\sigma_-^B + G(t)\sigma_-^A\sigma_z^B \\ &+ i\left(\int_0^t ds' P(t,s')z_{s'}\right)\sigma_-^A\sigma_-^B,\end{aligned} \tag{6.28}$$

where

$$\begin{aligned} A(t) &\equiv \int_0^t ds\alpha(t,s)a(t,s), \\ B(t) &\equiv \int_0^t ds\alpha(t,s)b(t,s), \\ F(t) &\equiv \int_0^t ds\alpha(t,s)f(t,s), \\ G(t) &\equiv \int_0^t ds\alpha(t,s)g(t,s), \\ P(t,s') &\equiv \int_0^t ds\alpha(t,s)p(t,s,s'). \end{aligned}$$

It can be checked that Eq. (6.27) indeed provides a consistent solution to Eq. (6.26). By substituting Eq. (6.27) into Eq. (6.26), the left-hand side (LHS) of it expands to

$$\begin{aligned}\frac{\partial\hat{\mathcal{O}}(t,s,z)}{\partial t} =& \frac{\partial a(t,s)}{\partial t}\sigma_-^A + \frac{\partial b(t,s)}{\partial t}\sigma_-^B + \frac{\partial f(t,s)}{\partial t}\sigma_z^A\sigma_-^B + \frac{\partial g(t,s)}{\partial t}\sigma_-^A\sigma_z^B \\ &+ ip(t,s,t)z_t + i\left(\int_0^t ds'\frac{\partial p(t,s,s')}{\partial t}z_{s'}\right)\sigma_-^A\sigma_-^B,\end{aligned}$$

while the right-hand side (RHS) of Eq. (6.26) is composed of the following commutators:

$$[\frac{-i\omega_A}{2}\sigma_z^A,\hat{\mathcal{O}}(t,s,z)] = \quad i\omega_A[a(t,s)\sigma_-^A + g(t,s)\sigma_-^A\sigma_z^B + i\left(\int_0^t ds'p(t,s,s')z_{s'}\right)\sigma_-^A\sigma_-^B],$$

$$[\frac{-i\omega_B}{2}\sigma_z^B,\hat{\mathcal{O}}(t,s,z)] = \quad i\omega_B[b(t,s)\sigma_-^B + f(t,s)\sigma_z^A\sigma_-^B + i\left(\int_0^t ds'p(t,s,s')z_{s'}\right)\sigma_-^A\sigma_-^B],$$

$$[\mathcal{L}z_t,\hat{\mathcal{O}}(t,s,z)] = 2z_t[f(t,s) + \kappa g(t,s)],$$

$$\begin{aligned}-[\sigma_+^A\bar{\mathcal{O}}(t,z),\hat{\mathcal{O}}(t,s,z)] =& A(t)[a(t,s)\sigma_-^A + g(t,s)\sigma_-^A\sigma_z^B] \\ &- B(t)[a(t,s)\sigma_z^A\sigma_-^B + g(t,s)\sigma_-^B] \\ &+ F(t)[a(t,s)\sigma_z^A\sigma_-^B + g(t,s)\sigma_-^B] \\ &+ G(t)[a(t,s)\sigma_-^A\sigma_z^B + g(t,s)\sigma_-^A \\ &+ (b(t,s) + f(t,s))(\sigma_-^B + \sigma_z^A\sigma_-^B)] \\ &+ i(A(t) + G(t))\left(\int_0^t ds'p(t,s,s')z_{s'}\right)\sigma_-^A\sigma_-^B \\ &- i\int_0^t ds'P(t,s')z_{s'}[g(t,s) - a(t,s)]\sigma_-^A\sigma_-^B,\end{aligned}$$

$$
\begin{aligned}
-\kappa[\sigma_+^B\bar{\mathcal{O}}(t,z),\hat{\mathcal{O}}(t,s,z)] = &-\kappa A(t)[f(t,s)\sigma_-^A + b(t,s)\sigma_-^A\sigma_z^B] \\
&+\kappa B(t)[b(t,s)\sigma_-^B + f(t,s)\sigma_z^A\sigma_-^B] \\
&+\kappa F(t)[(a(t,s)+g(t,s))(\sigma_-^A+\sigma_-^A\sigma_z^B)] \\
&b(t,s)\sigma_z^A\sigma_-^B + f(t,s)\sigma_-^B] \\
&+\kappa G(t)[b(t,s)\sigma_-^A\sigma_z^B + f(t,s)\sigma_-^A] \\
&+i\kappa(B(t)+F(t))\int_0^t ds' p(t,s,s')z_{s'}\sigma_-^A\sigma_-^B \\
&+i\kappa\left(\int_0^t ds' P(t,s')z_{s'}\right)[b(t,s)-f(t,s)]\sigma_-^A\sigma_-^B,
\end{aligned}
$$

as well as

$$
\begin{aligned}
-\mathcal{L}^\dagger\frac{\delta\bar{\mathcal{O}}(t,z)}{\delta z_s} &= -(\sigma_+^A+\kappa\sigma_+^B)\frac{\delta[i\int_0^t ds' P(t,s')z_{s'}]}{\delta z_s}\sigma_-^A\sigma_-^B \\
&= -iP(t,s)[\frac{1}{2}(\sigma_z^A\sigma_-^B+\sigma_-^B)+\frac{\kappa}{2}(\sigma_-^A+\sigma_-^A\sigma_z^B)].
\end{aligned}
$$

By equating the LHS with the RHS, we obtain the following partial differential equations for the coefficient functions $a(t,s), b(t,s), f(t,s), g(t,s)$ and $p(t,s,s')$:

$$
\begin{aligned}
\frac{\partial a(t,s)}{\partial t} &= i\omega_A a(t,s) + A(t)a(t,s) + G(t)g(t,s) \\
&+\kappa F(t)[a(t,s)+g(t,s)] + \kappa[G(t)-A(t)]f(t,s) - \frac{i\kappa}{2}P(t,s),
\end{aligned}
\tag{6.29}
$$

$$
\begin{aligned}
\frac{\partial b(t,s)}{\partial t} &= i\omega_B b(t,s) + \kappa B(t)b(t,s) + \kappa F(t)f(t,s) \\
&+G(t)[b(t,s)+f(t,s)] + [F(t)-B(t)]g(t,s) - \frac{i}{2}P(t,s),
\end{aligned}
\tag{6.30}
$$

$$\frac{\partial f(t,s)}{\partial t} = i\omega_B f(t,s) + \kappa F(t) b(t,s) + \kappa B(t) f(t,s)$$
$$+ G(t)[b(t,s) + f(t,s)] + [F(t) - B(t)]a(t,s) - \frac{i}{2}P(t,s), \qquad (6.31)$$

$$\frac{\partial g(t,s)}{\partial t} = i\omega_A g(t,s) + G(t)a(t,s) + A(t)g(t,s)$$
$$+ \kappa F(t)[a(t,s) + g(t,s)] + \kappa[G(t) - A(t)]b(t,s) - \frac{i\kappa}{2}P(t,s), \qquad (6.32)$$

$$\frac{\partial p(t,s,s')}{\partial t} = i(\omega_A + \omega_B)p(t,s,s') + [A(t) + G(t) + \kappa B(t) + \kappa F(t)]p(t,s,s')$$
$$+ P(t,s')[a(t,s) - g(t,s) + \kappa b(t,s) - \kappa f(t,s)], \qquad (6.33)$$

as well as the boundary condition

$$p(t,s,t) = -2if(t,s) - 2i\kappa g(t,s). \qquad (6.34)$$

We also deduce the initial conditions $a(s,s) = 1$, $b(s,s) = \kappa$, and $f(s,s) = g(s,s) = p(s,s,s') = 0$ from the fact that $\hat{\mathcal{O}}(s,s,z) = \mathcal{L}$.

Eqs. (6.29-6.33) are the required exact equations that govern the O-operator evolution and, in principle, allow us to numerically solve the QSD equation. We now consider the bath correlation function to be an Ornstein-Uhlenbeck process such that $\alpha(t,s) = \frac{\gamma}{2}e^{-\gamma|t-s|}$, which facilitates a set of simpler ordinary differential equations from the above Eqs. (6.29-6.33), as presented in Eq. (9). For instance, we have

$$\begin{aligned}\frac{\partial A(t)}{\partial t} &= \frac{\partial}{\partial t}\int_0^t ds\alpha(t,s)a(t,s) \\ &= \alpha(t,t)a(t,t) + \int_0^t ds\frac{\partial\alpha(t,s)}{\partial t}a(t,s) + \int_0^t ds\alpha(t,s)\frac{\partial a(t,s)}{dt} \\ &= \frac{\gamma}{2} - \gamma\int_0^t ds\alpha(t,s)a(t,s) + \int_0^t ds\alpha(t,s)[i\omega_A a(t,s) + A(t)a(t,s) \\ &+ G(t)g(t,s) + \kappa F(t)(a(t,s)+g(t,s)) + \kappa(G(t)-A(t))f(t,s) - \frac{i\kappa}{2}P(t,s)] \\ &= \frac{\gamma}{2} - \gamma A(t) + i\omega_A A(t) + A^2(t) + G^2(t) + \kappa F(t)G(t) - \frac{i\kappa}{2}Q(t),\end{aligned} \tag{6.35}$$

where $Q(t) \equiv \int_0^t ds\alpha(t,s)P(t,s)$. It is noted that the initial condition $a(s,s)=1$ has been used. Applying a similar derivation, we also have

$$\frac{\partial B(t)}{\partial t} = \frac{\gamma\kappa}{2} - \gamma B(t) + i\omega_B B(t) + \kappa B^2(t) + \kappa F^2(t) + 2G(t)F(t) - \frac{i}{2}Q(t), \tag{6.36}$$

$$\begin{aligned}\frac{\partial F(t)}{\partial t} =& - \gamma F(t) + i\omega_B F(t) + 2\kappa B(t)F(t) + B(t)G(t) + F(t)G(t) \\ &+ A(t)F(t) - A(t)B(t) - \frac{i}{2}Q(t),\end{aligned} \tag{6.37}$$

$$\begin{aligned}\frac{\partial G(t)}{\partial t} =& - \gamma G(t) + i\omega_A G(t) + 2A(t)G(t) + \kappa A(t)F(t) + \kappa F(t)G(t) \\ &+ \kappa B(t)G(t) - \kappa A(t)B(t) - \frac{i\kappa}{2}Q(t),\end{aligned} \tag{6.38}$$

and

$$\frac{\partial P(t,s')}{\partial t} = -\gamma P(t,s') + i(\omega_A+\omega_B)P(t,s') + [2A(t) + 2\kappa B(t)]P(t,s'). \tag{6.39}$$

By the boundary condition in Eq. (6.34) and the definitions of $P(t,s')$, $F(t)$, and $G(t)$, it is found $P(t,t) = -2iF(t) - 2i\kappa G(t)$ and the solution of Eq. (6.39) is Eq. (5.25).

To construct a closed group of differential equations for Eqs. (6.35-6.38), we derive the ordinary differential equation for $Q(t)$ using Eq. (6.39):

$$\begin{aligned}
\frac{\partial Q(t)}{\partial t} &= \frac{\partial}{\partial t}\int_0^t ds'\alpha(t,s')P(t,s') \\
&= \alpha(t,t)P(t,t) + \int_0^t ds'\frac{\partial\alpha(t,s')}{\partial t}P(t,s') + \int_0^t ds'\alpha(t,s)\frac{\partial P(t,s')}{dt} \\
&= \frac{\gamma}{2}[-2iF(t) - 2i\kappa G(t)] - \gamma\int_0^t ds'\alpha(t,s')P(t,s') \\
&+ \int_0^t ds'\alpha(t,s')[-\gamma P(t,s') + i(\omega_A+\omega_B)P(t,s') + (2A(t)+2\kappa B(t))P(t,s')] \\
&= -i\gamma[F(t)+\kappa G(t)] - 2\gamma Q(t) + i(\omega_A+\omega_B)Q(t) + [2A(t)+2\kappa B(t)]Q(t).
\end{aligned} \tag{6.40}$$

By definition, the initial conditions for the coefficients of $\bar{\mathcal{O}}(t,z)$ are $A(0) = B(0) = F(0) = G(0) = Q(0) = 0$.

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George Washington University, Summer 2005